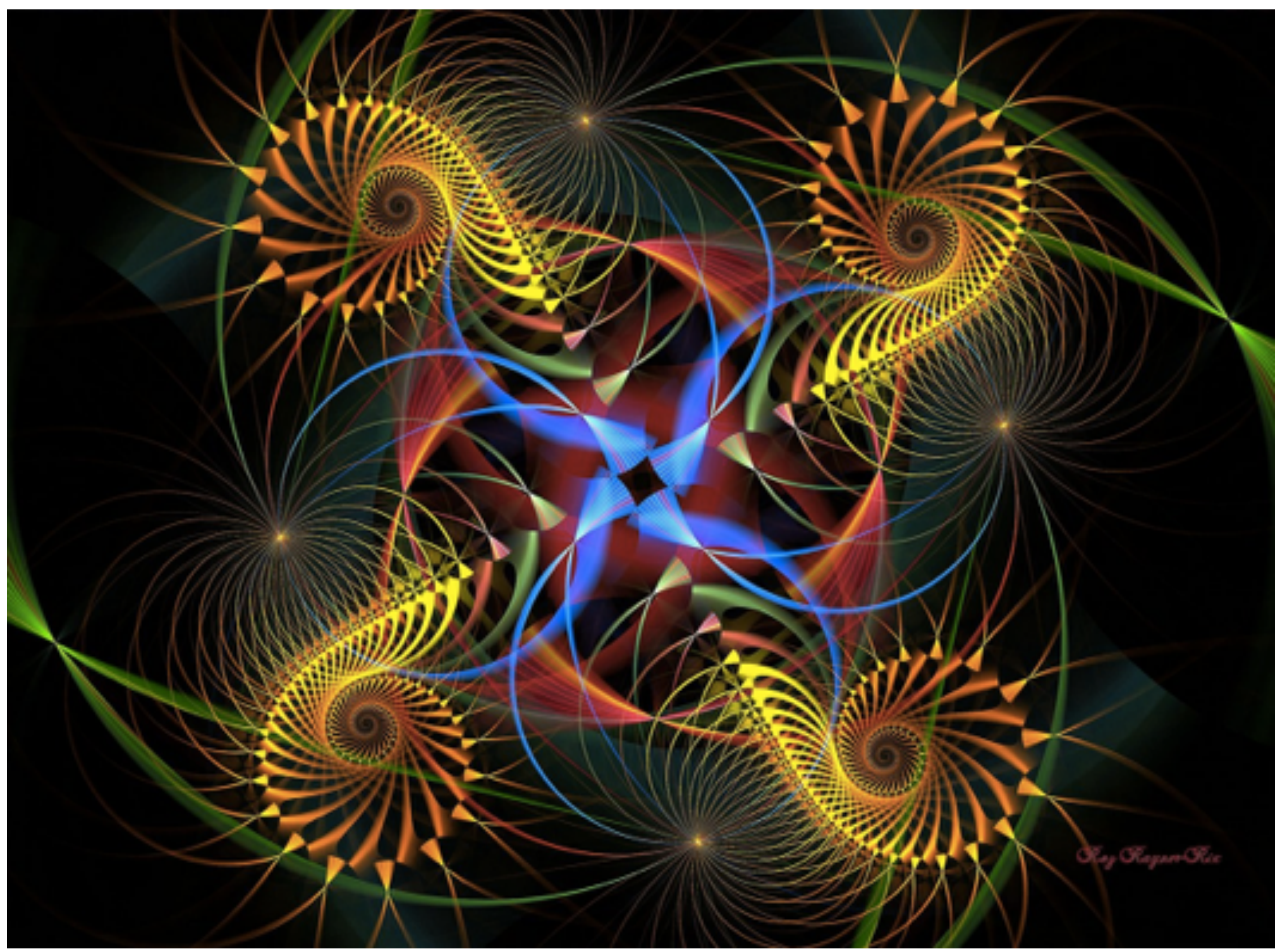
Introduction to Modern Geometry

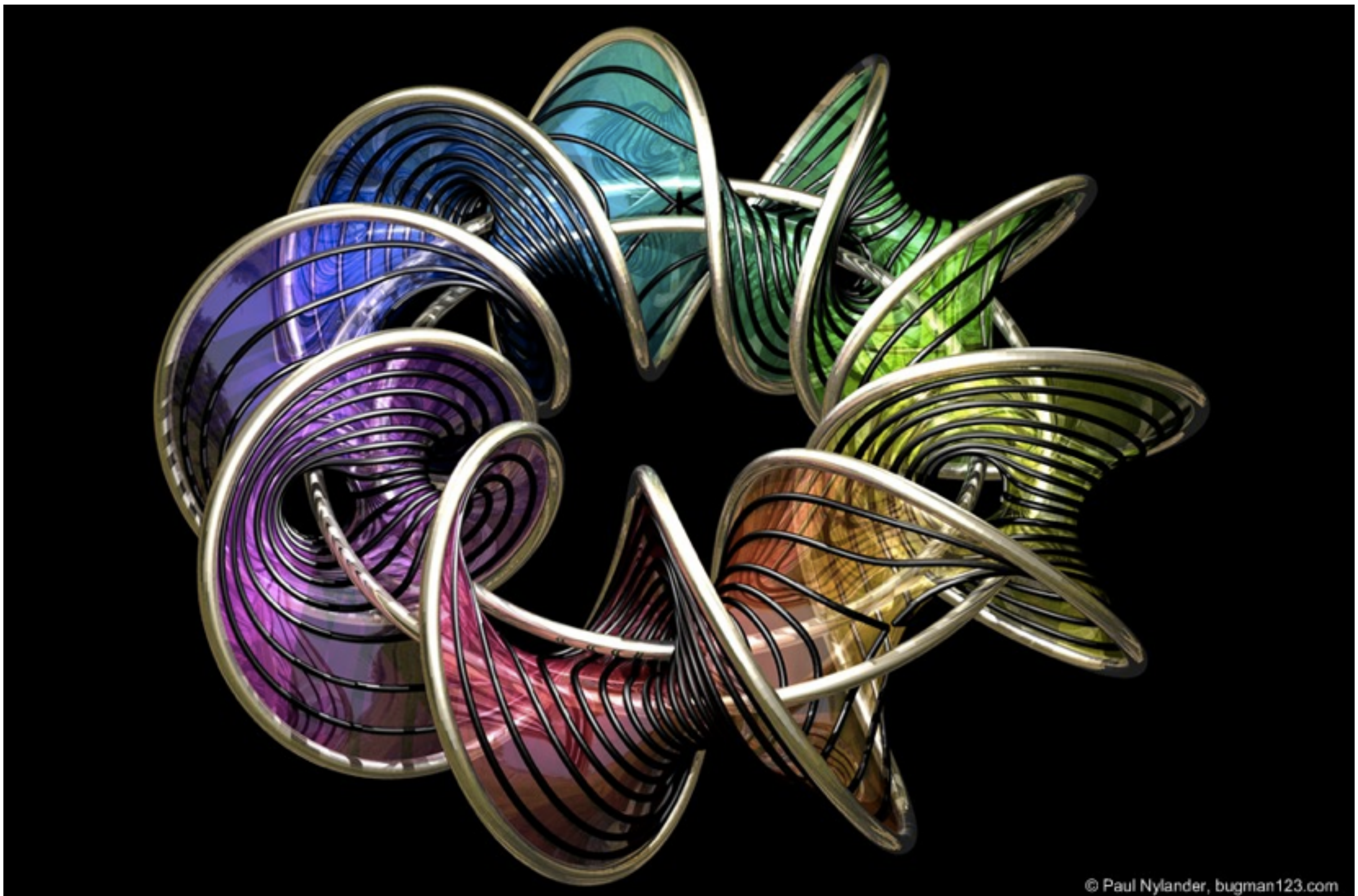
Splash 2015

Goal of talk:

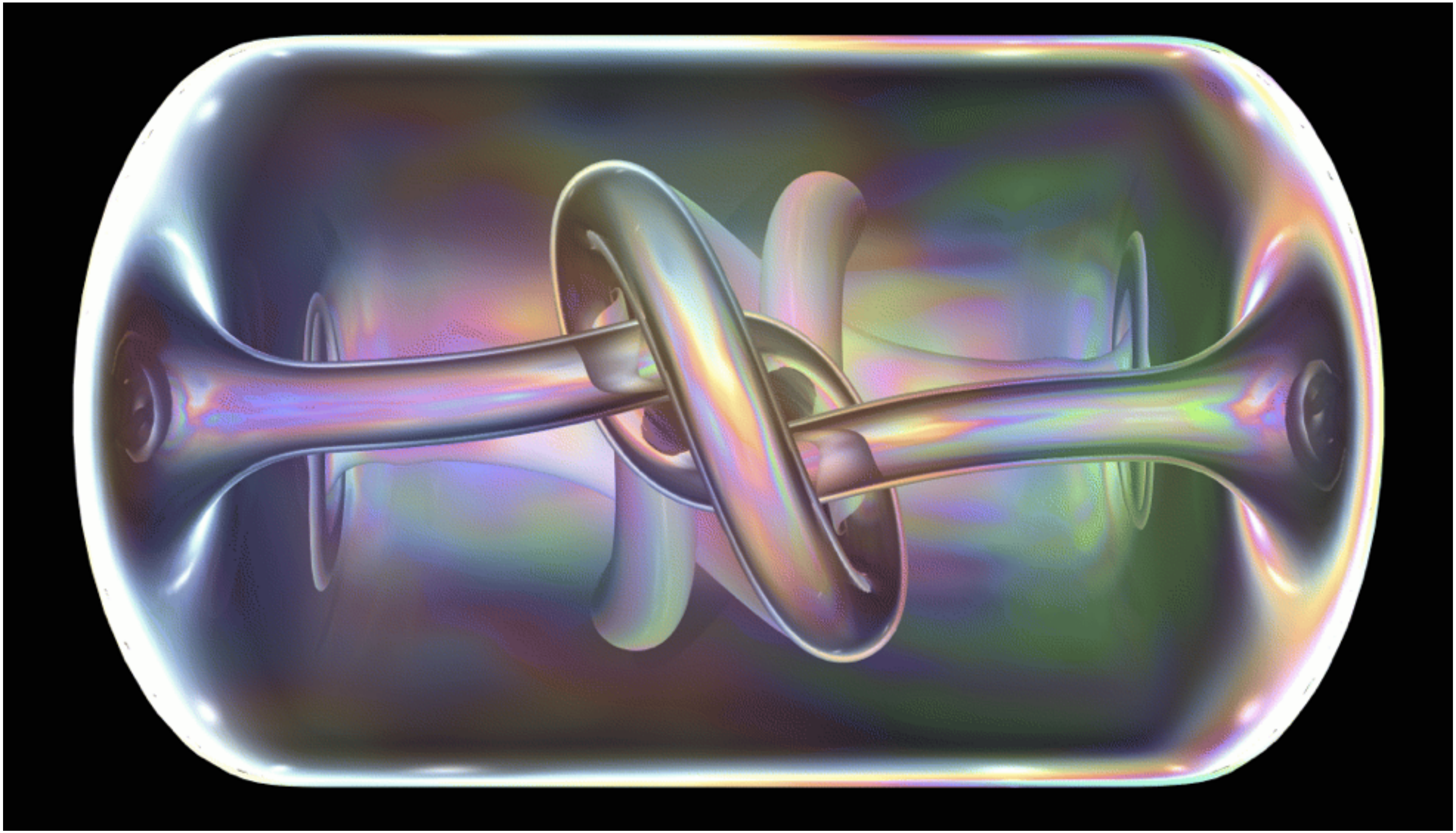
What is the “flavor” of current mathematics research, particularly in geometry?



Don’t worry if things are confusing (professional mathematicians also find them confusing!)



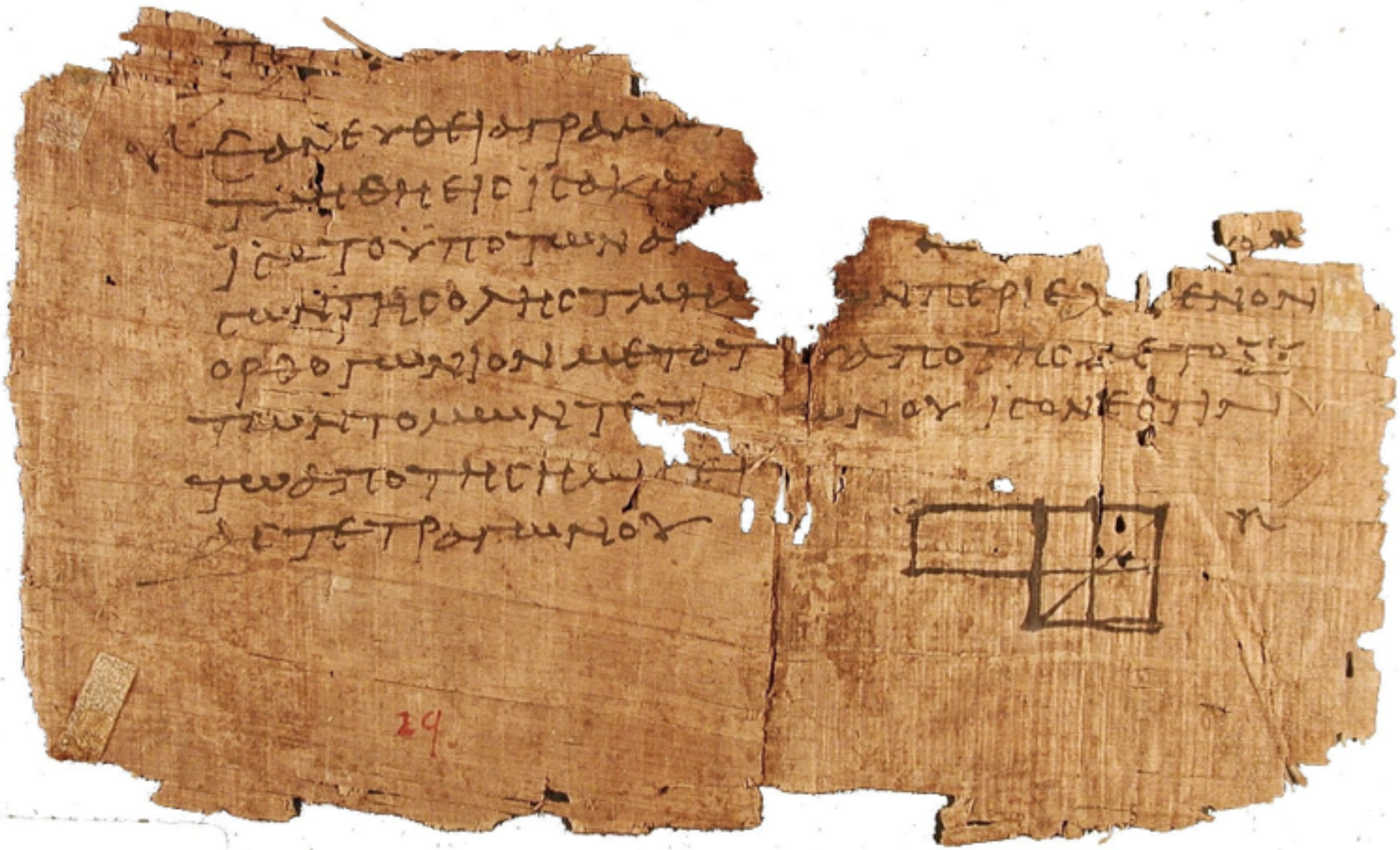
Please ask questions!



Ancient geometry:

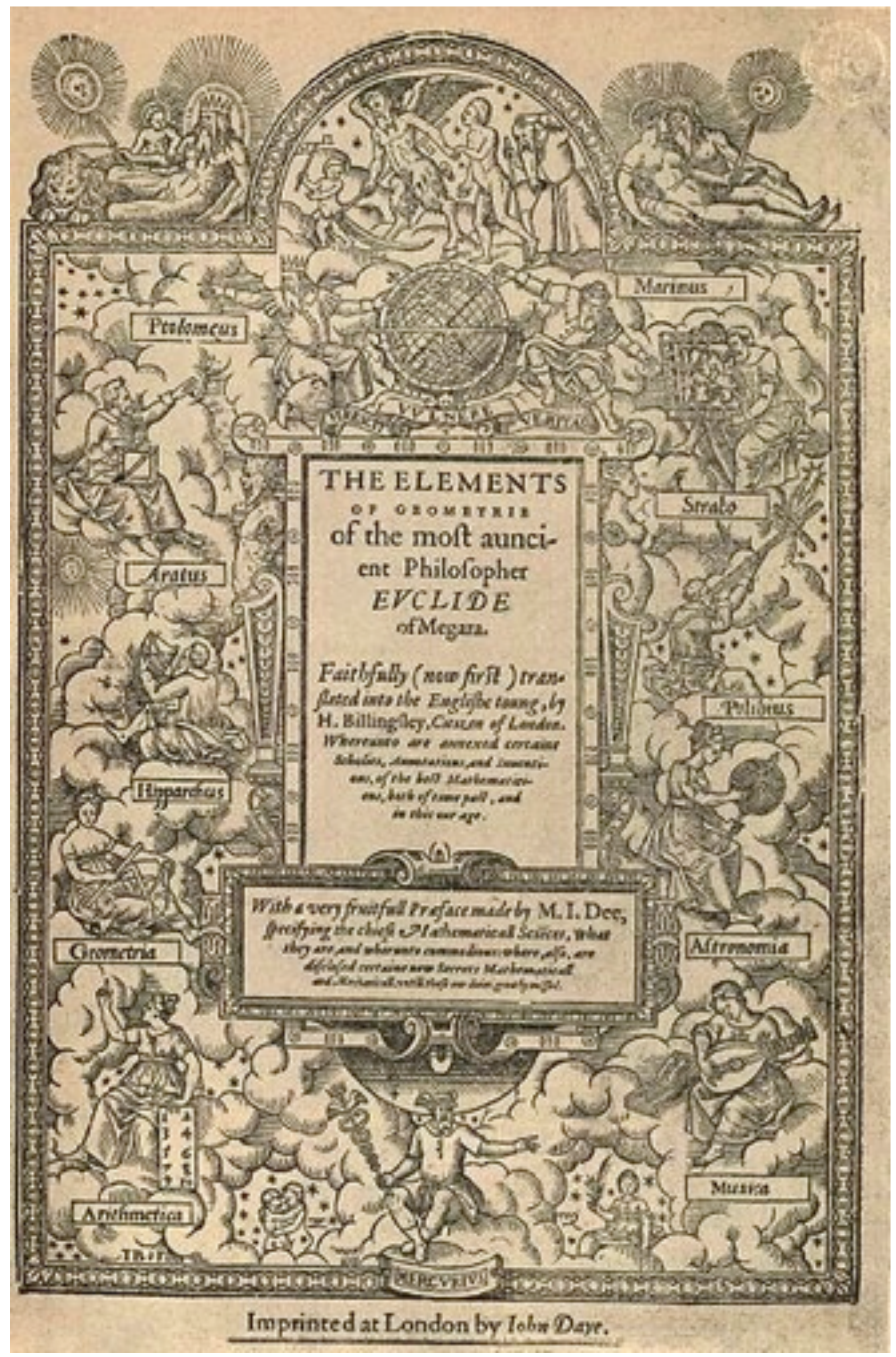
Euclid
~300 BC

Euclid’s “The Elements”



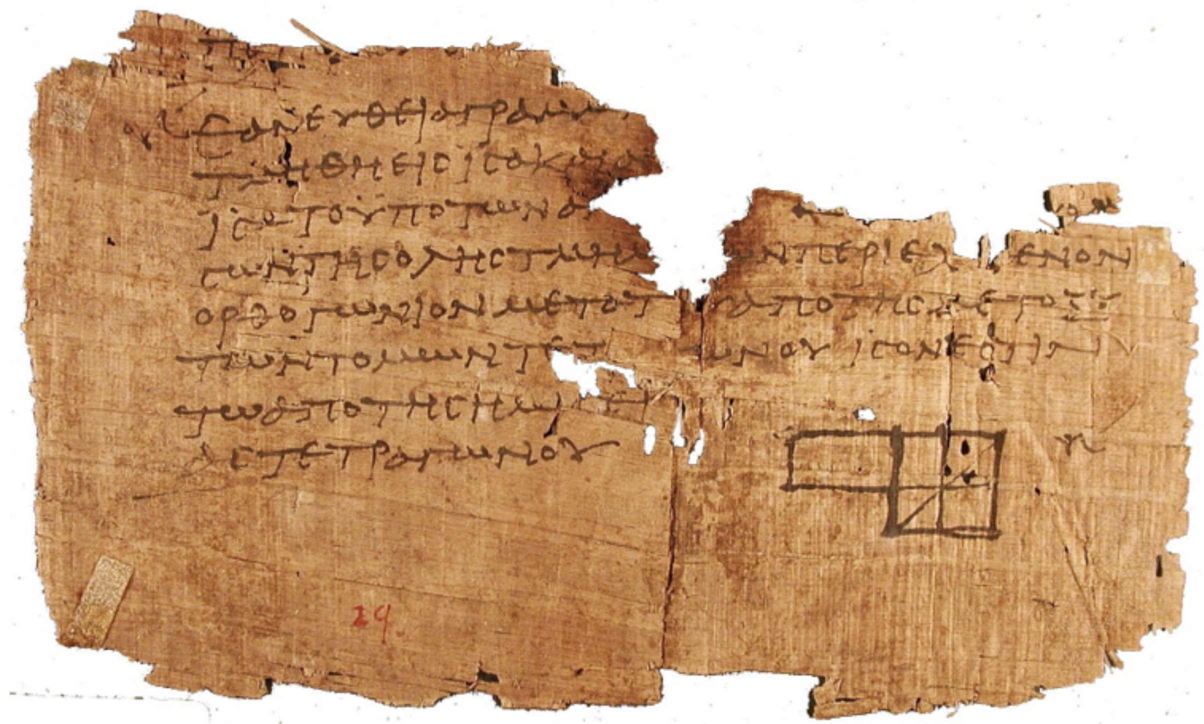
Papyrus from ~AD 75-125

Euclid’s “The Elements”



English version from 1570

Euclid's "The Elements"



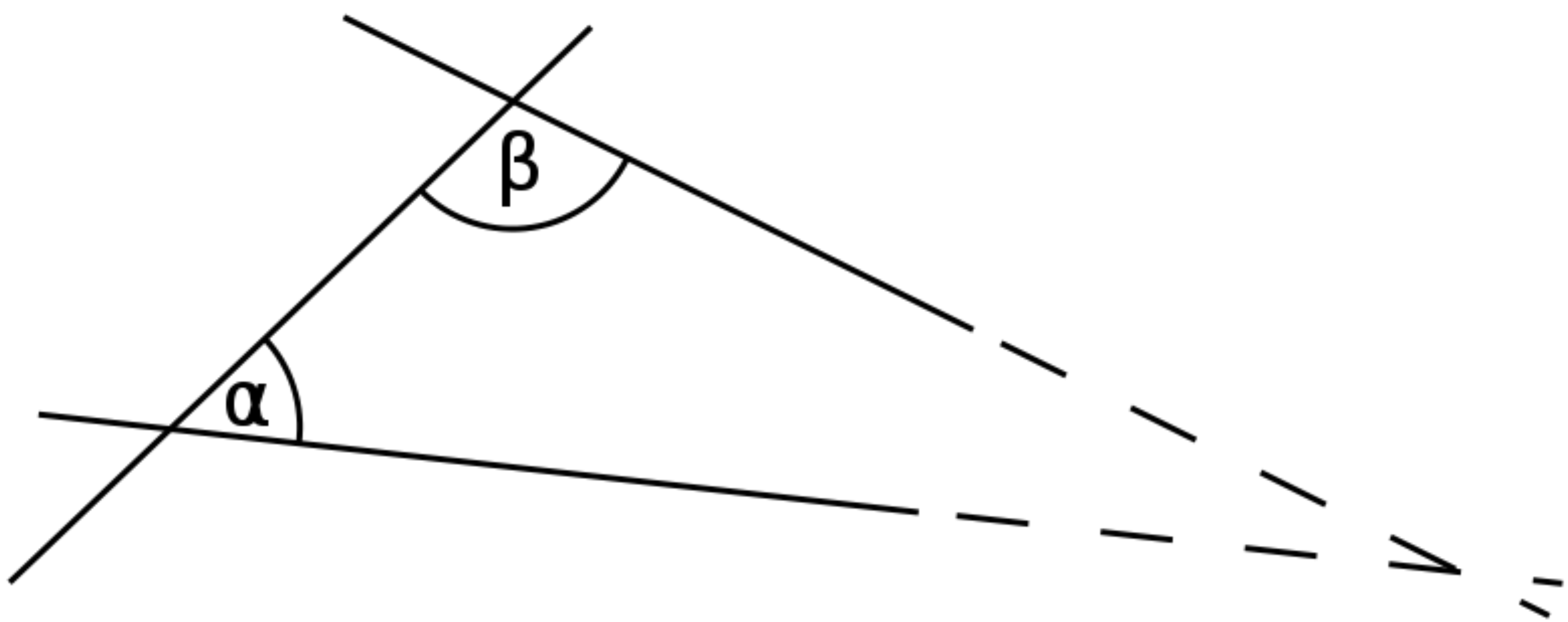
- Definitive reference on geometry for over two millennia!
- One of first examples of giving *rigorous* (irrefutable!) proofs basic on *postulates*
- *Postulate*: a fact which is so "basic" and "self-evident" that it cannot be proven from other facts, hence we must just *accept* that it has to be true

“Let the following be postulated”

1. "To draw a straight line from any point to any point.”
2. "To extend a finite straight line continuously in a straight line.”
3. "To describe a circle with any centre and radius.”
4. "That all right angles are equal to one another.”
5. The parallel postulate: "That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."

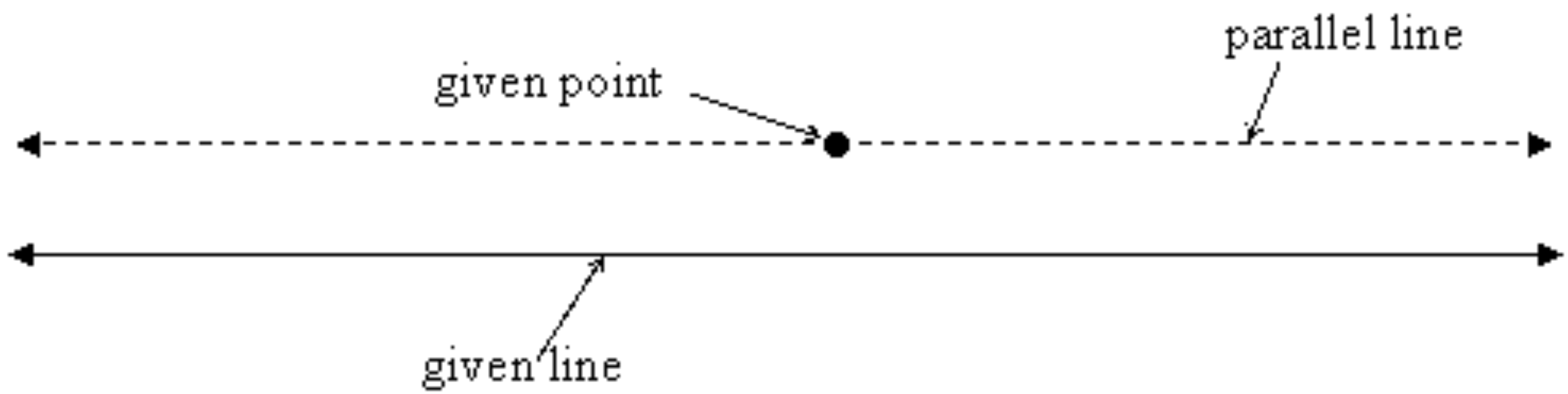
The troublesome one:

The parallel postulate: "That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."



An equivalent formulation of the Parallel Postulate:

"In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point."

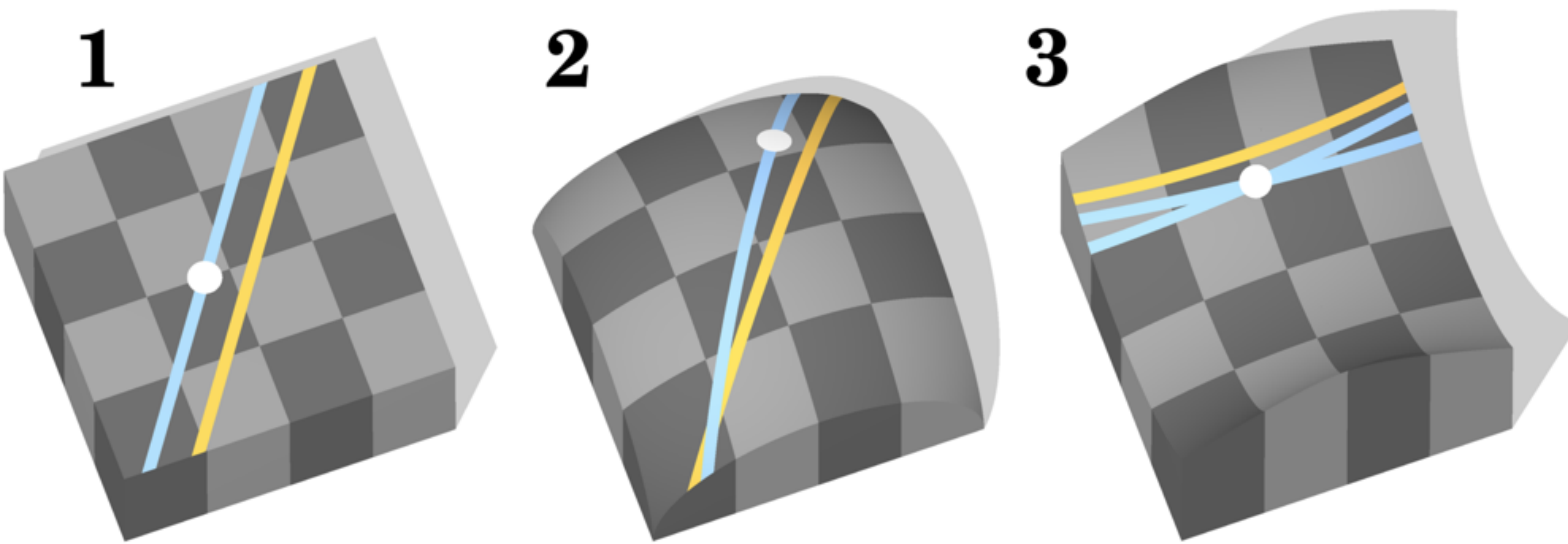


The Parallel Postulate: why so controversial?

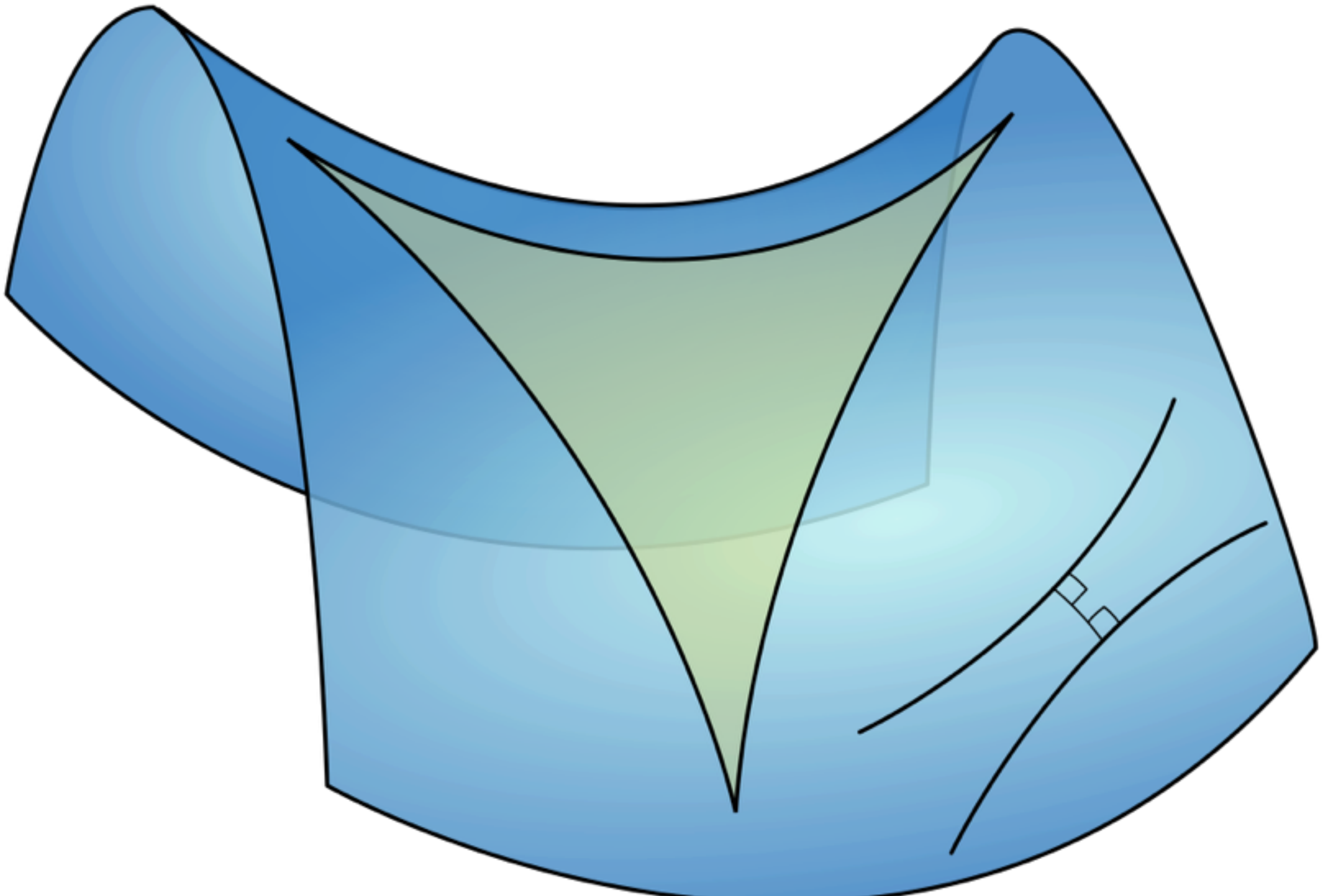
- it does not seem quite as “self-evident” as the other postulates
- people tried for thousands of years to prove that the parallel postulate in terms of the other postulates

Nikolai Lobachevsky, b. 1792

Turn out: the parallel postulate is not always true!

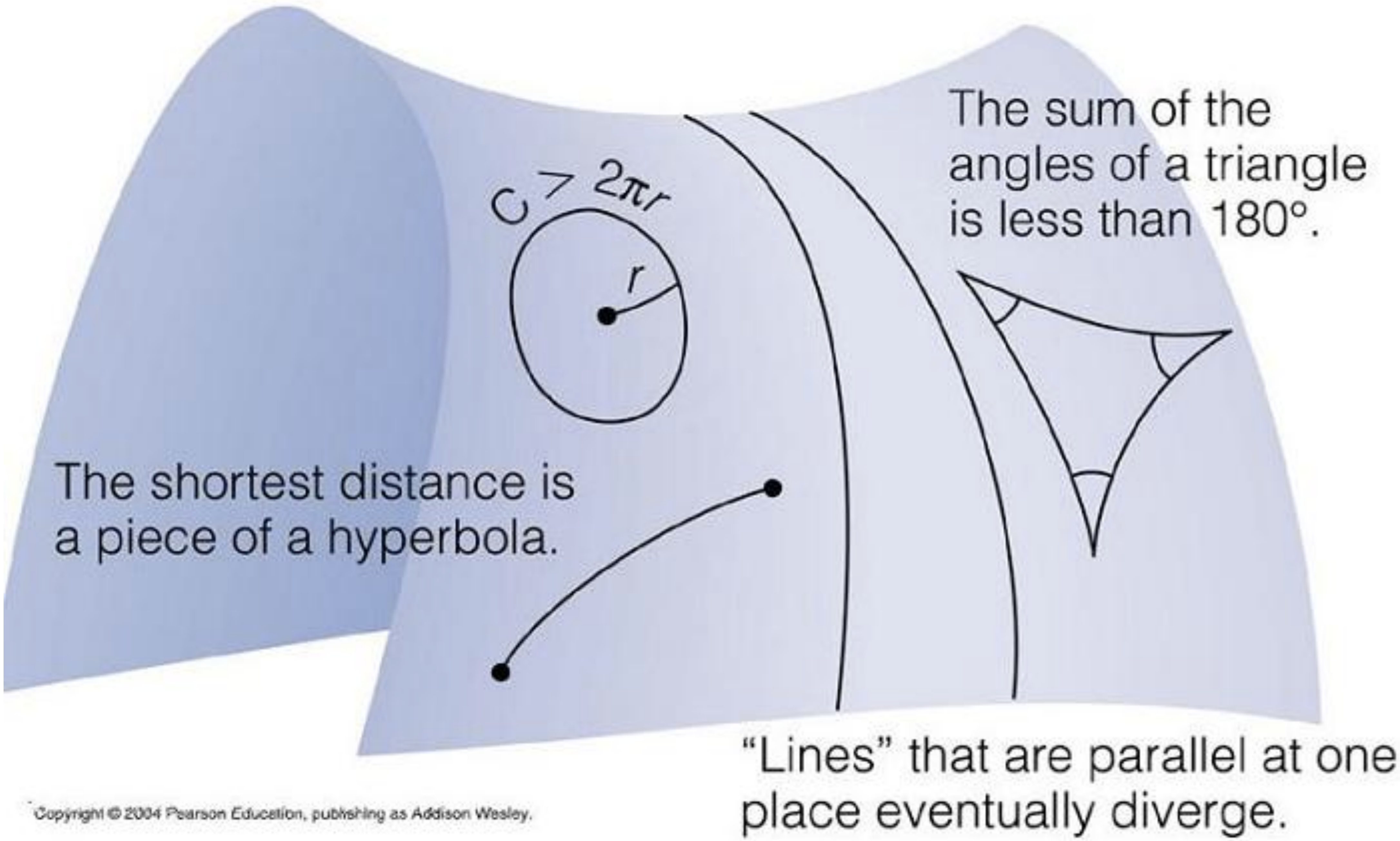


Hyperbolic geometry:

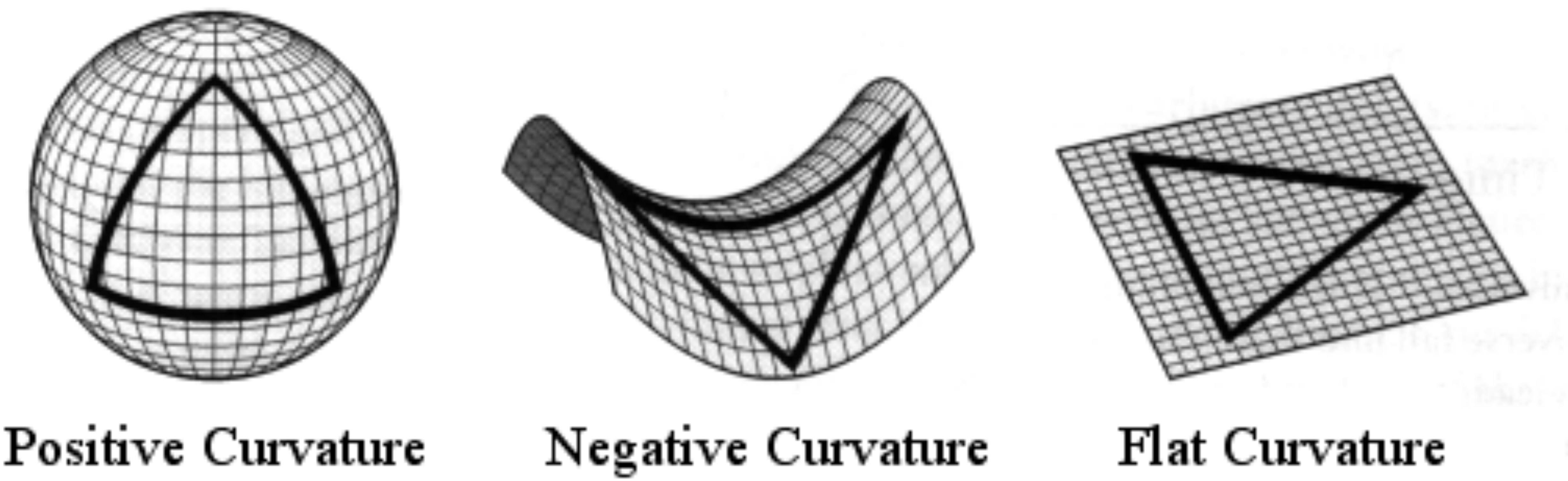


Hyperbolic geometry:

Hyperbolic geometry:



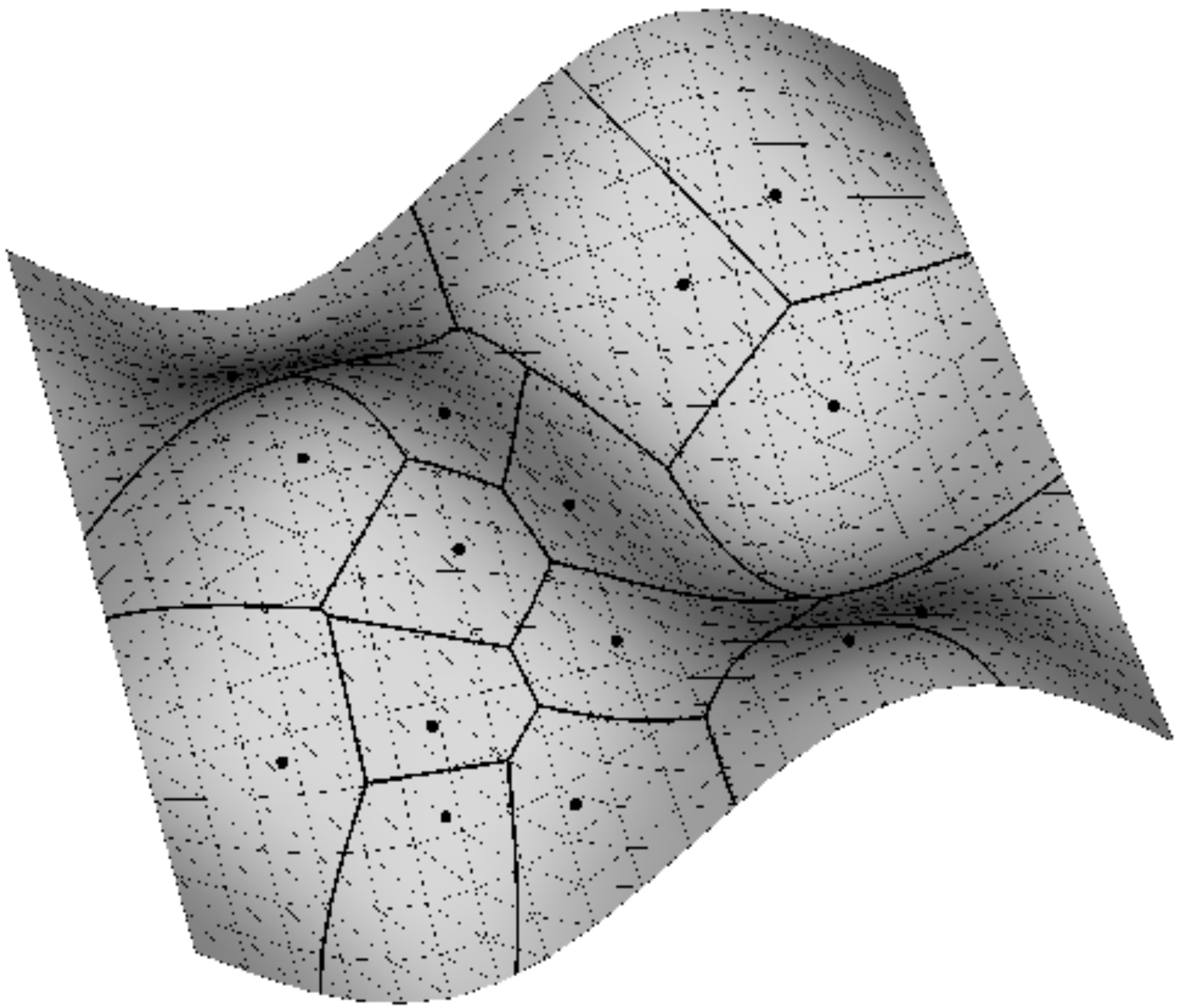
Related to notion of *curvature:*



Would later give rise to discovery of Riemannian geometry

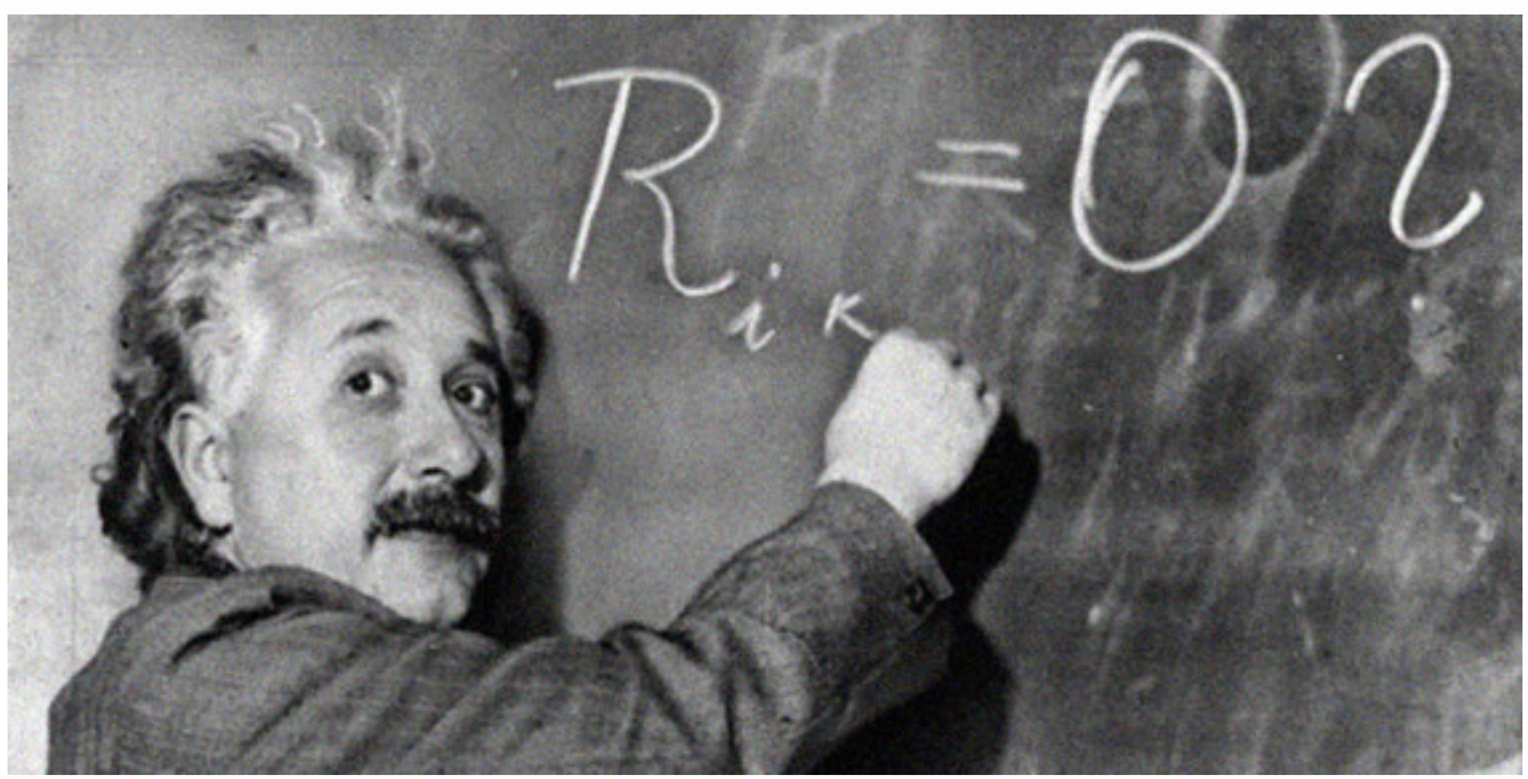
Bernhard Riemann, b.1826

Riemann discovered a mathematical framework in which the curvature could vary from point to point

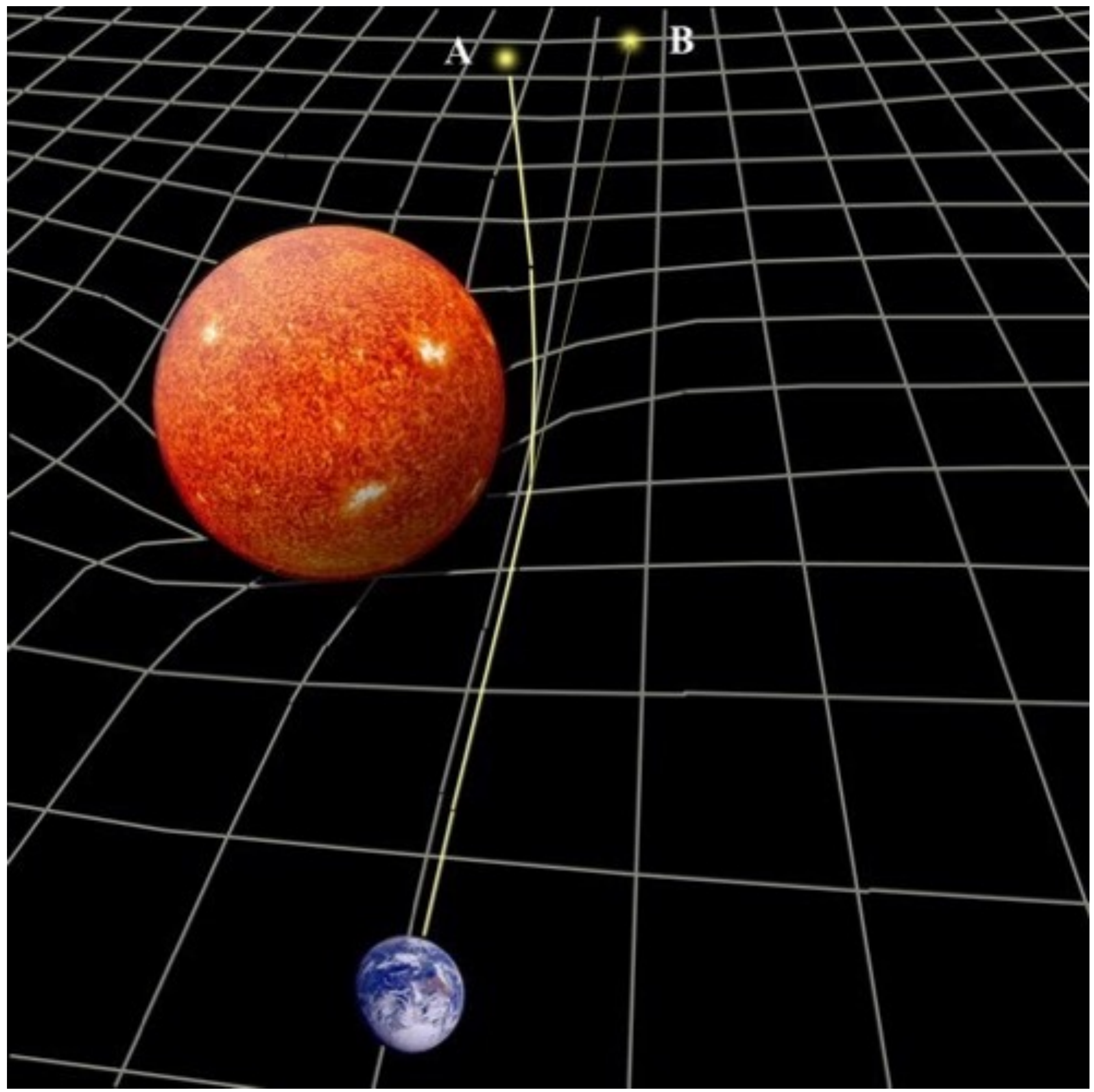


Riemannian geometry was used to Einstein as the mathematical basis of general relativity

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$



General relativity:



Topology:

to·pol·o·gy

/təˈpäləjē/

noun

noun: **topology**

1. MATHEMATICS

   the study of geometric properties and spatial relations unaffected by the continuous change of shape or size of figures.

   - a family of open subsets of an abstract space such that the union and the intersection of any two of them are members of the family, and that includes the space itself and the empty set.

     plural noun: **topologies**

2. the way in which constituent parts are interrelated or arranged.

   "the topology of a computer network"

Origin

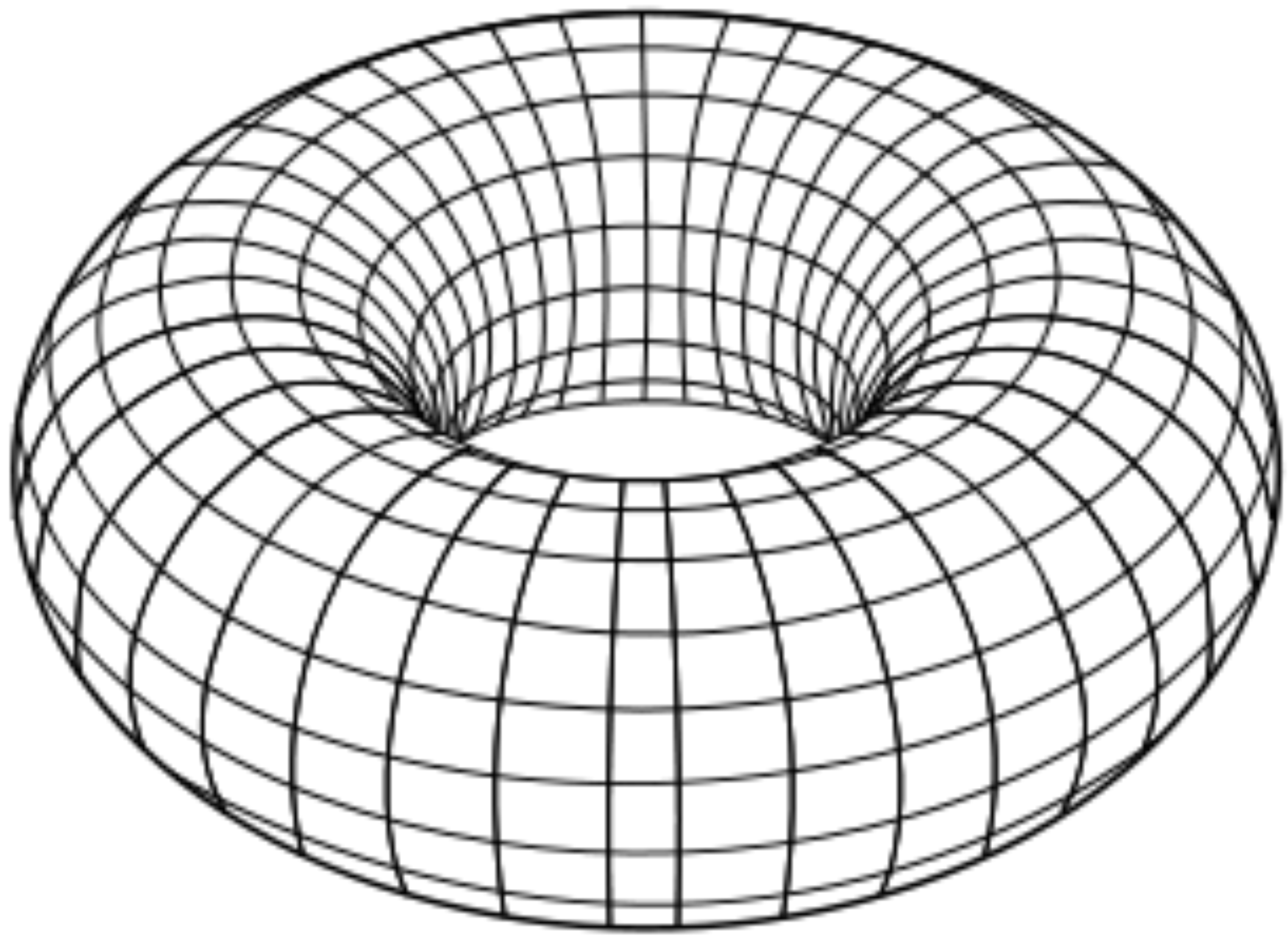
late 19th century: via German from Greek *topos* 'place' + -logy.

Topology:


Use over time for: topology
Mentions
1800
1850
1900
1950
2010

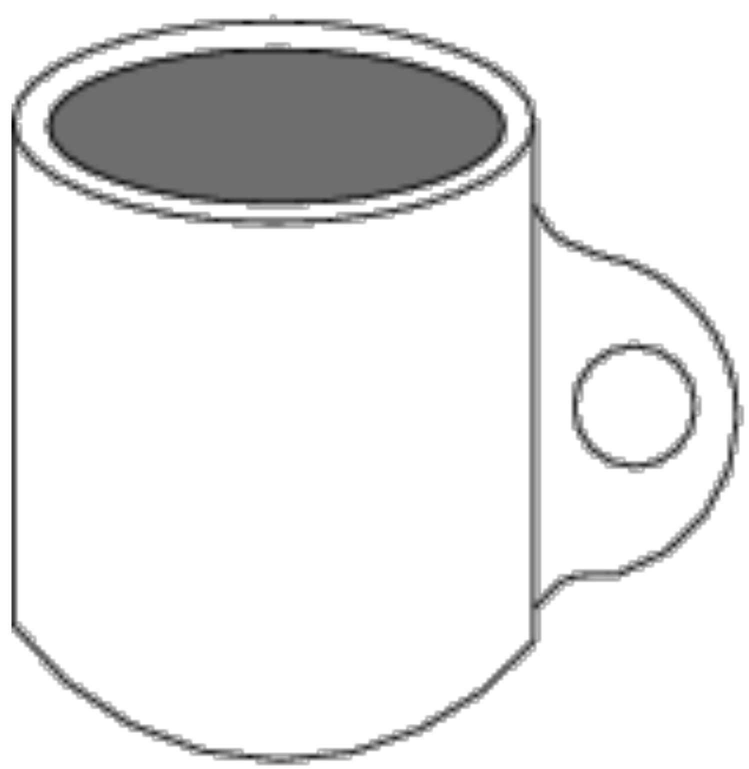
Topology:

“the study of geometric properties and spatial relations unaffected by the continuous change of shape or size of figures.”

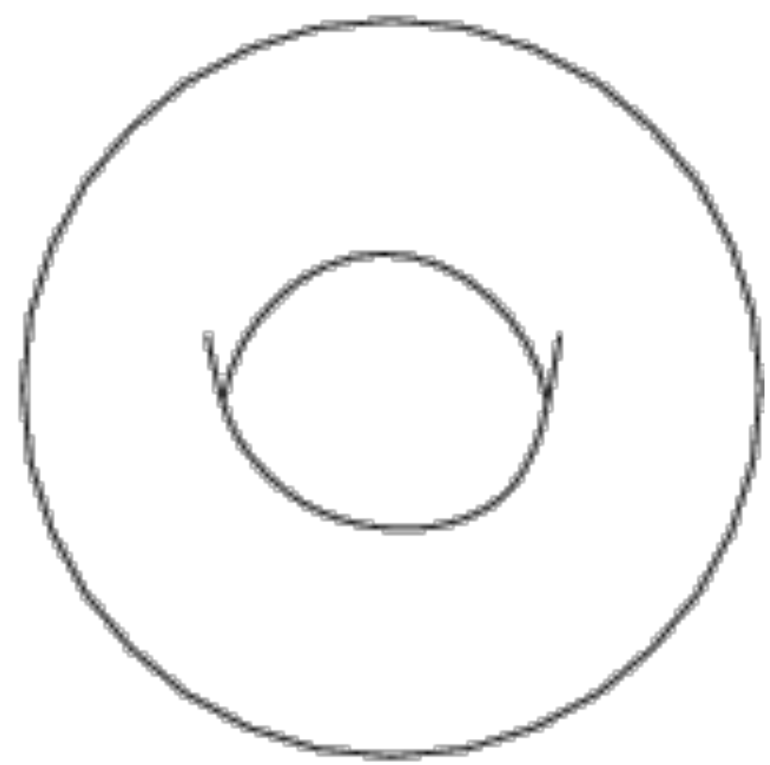


a “torus” (i.e. the surface of donut)

To a topologist, a coffee mug and a donut are the same thing!

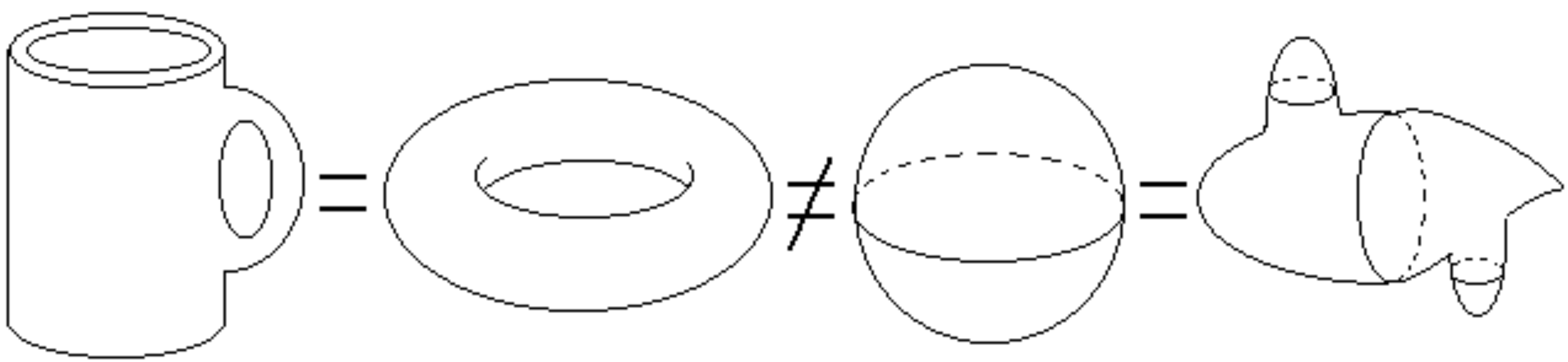


a torus

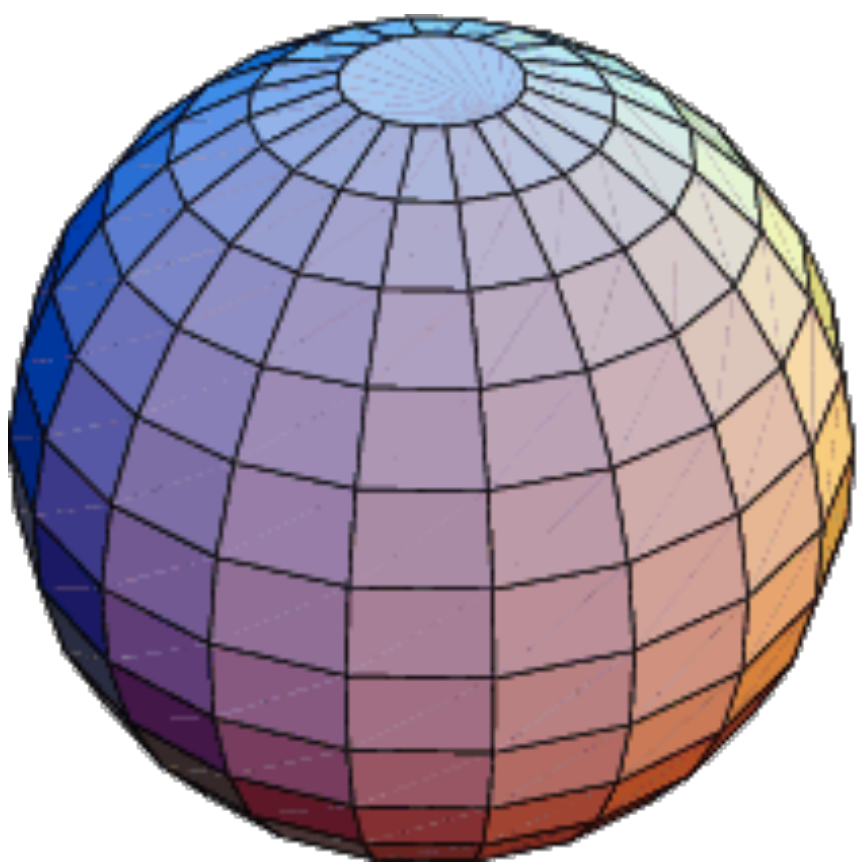


another torus

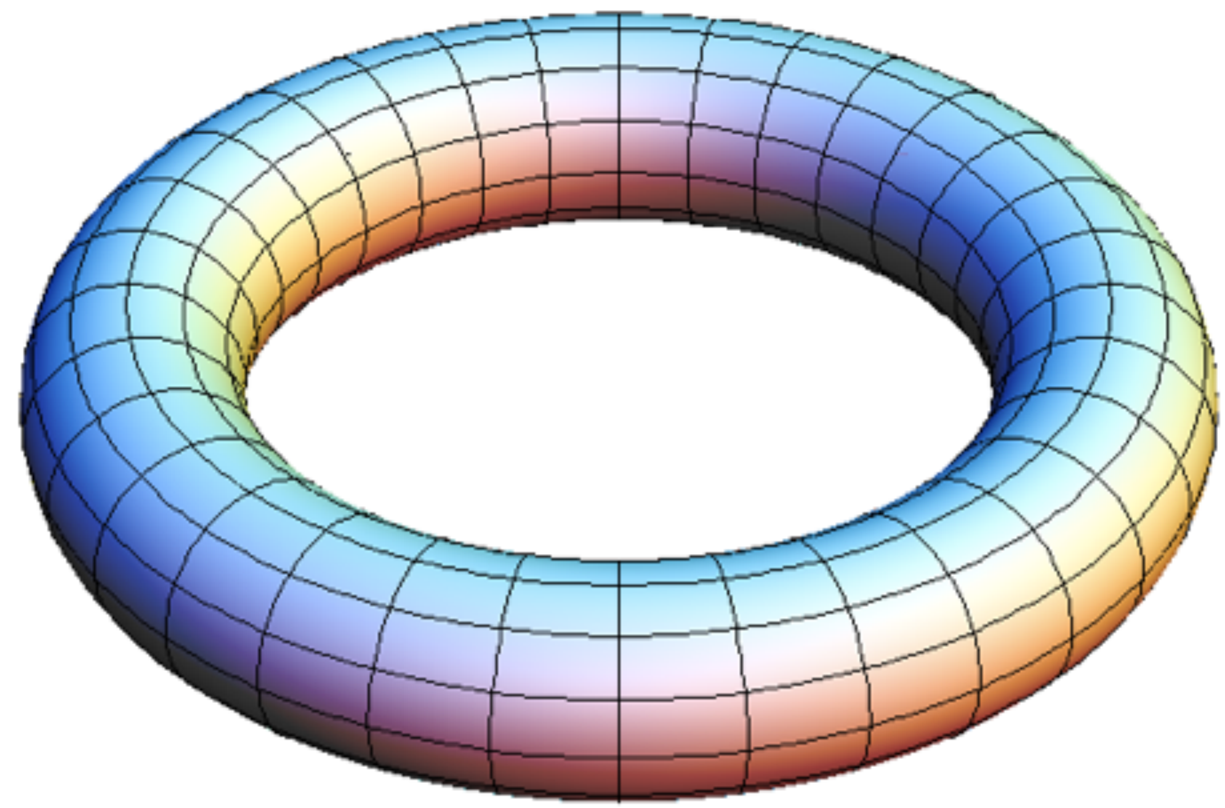
In topology shapes which can be continuously be deformed into each other are considered “equivalent”



Two spaces which are *not* topologically equivalent

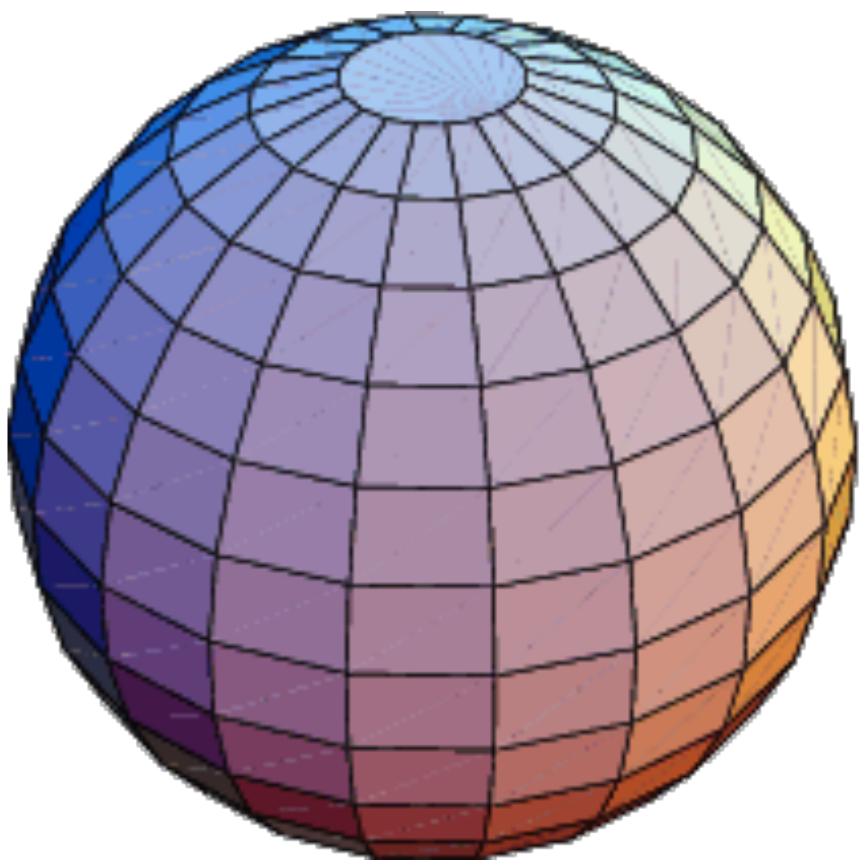


sphere

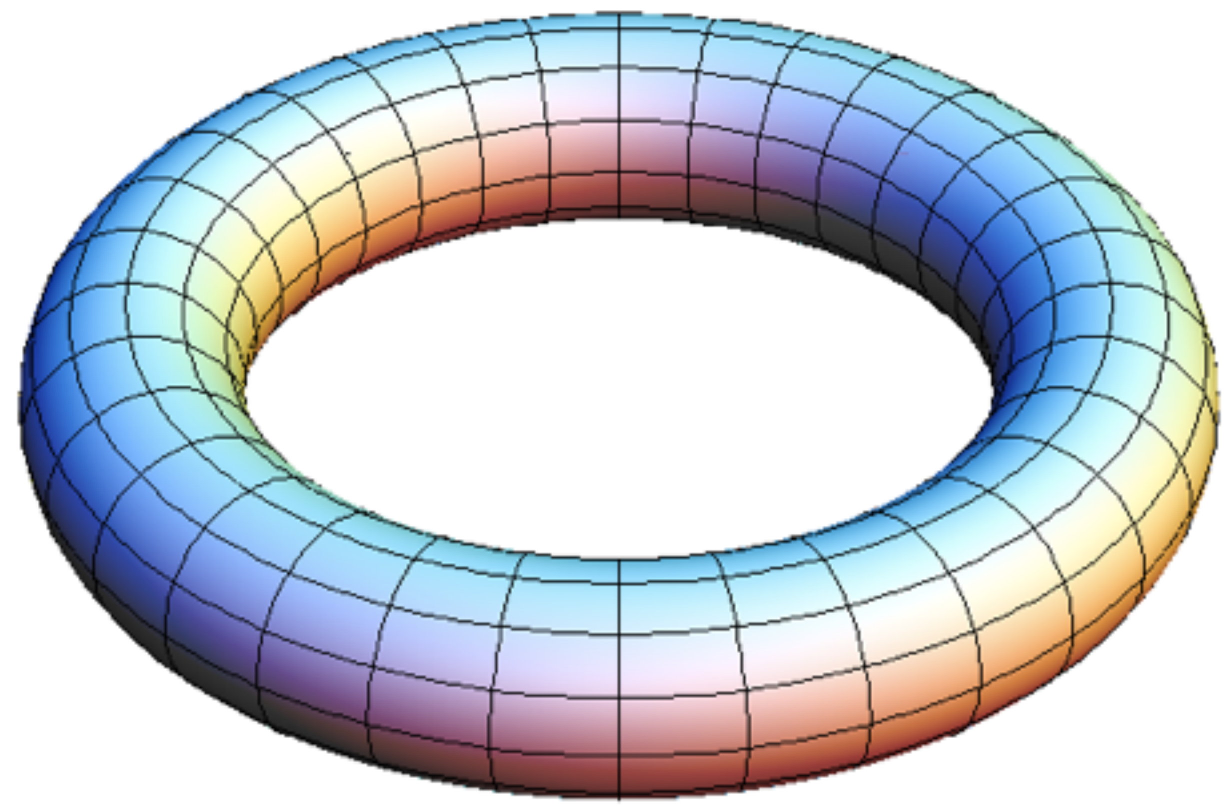


torus

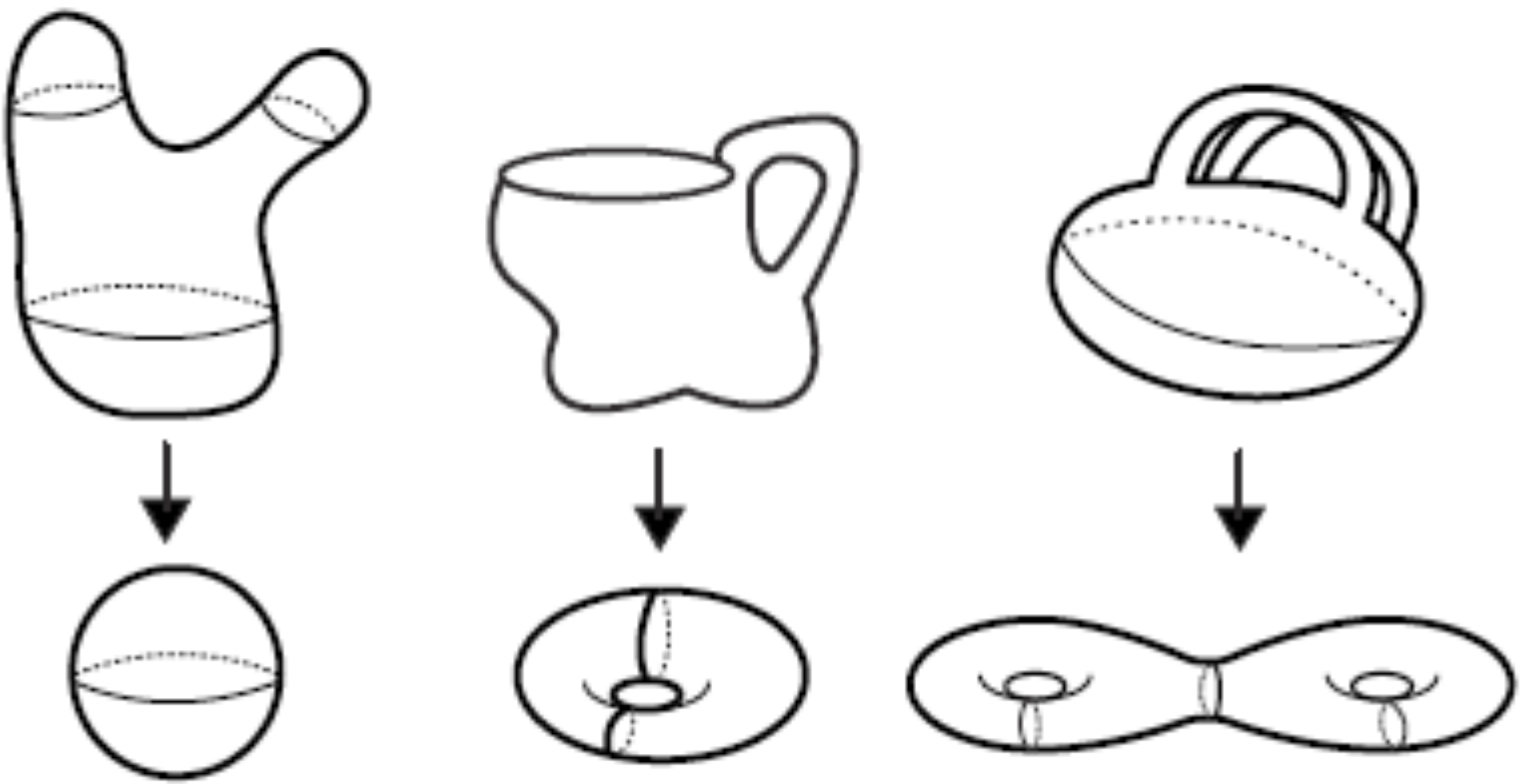
It is not possible to deform the sphere into the torus which causes “tears”!



sphere



torus

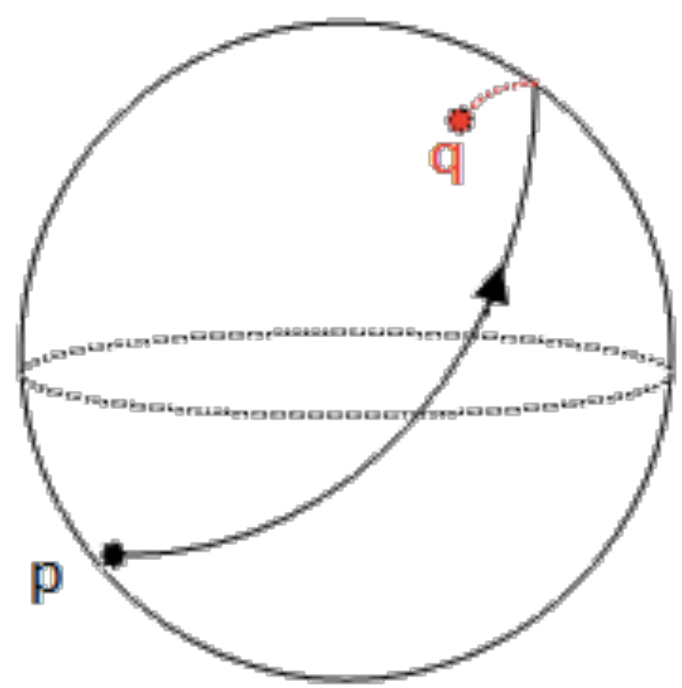


Example of a theorem in topology:

The Borsuk-Ulam theorem: given any two continuous functions: $f, g : S^2 \to \mathbb{R}$

there must an a pair of antipodal points $p, q \in S^2$ such that:

$$f(p) = g(p) \qquad \text{and} \qquad f(q) = g(q)$$



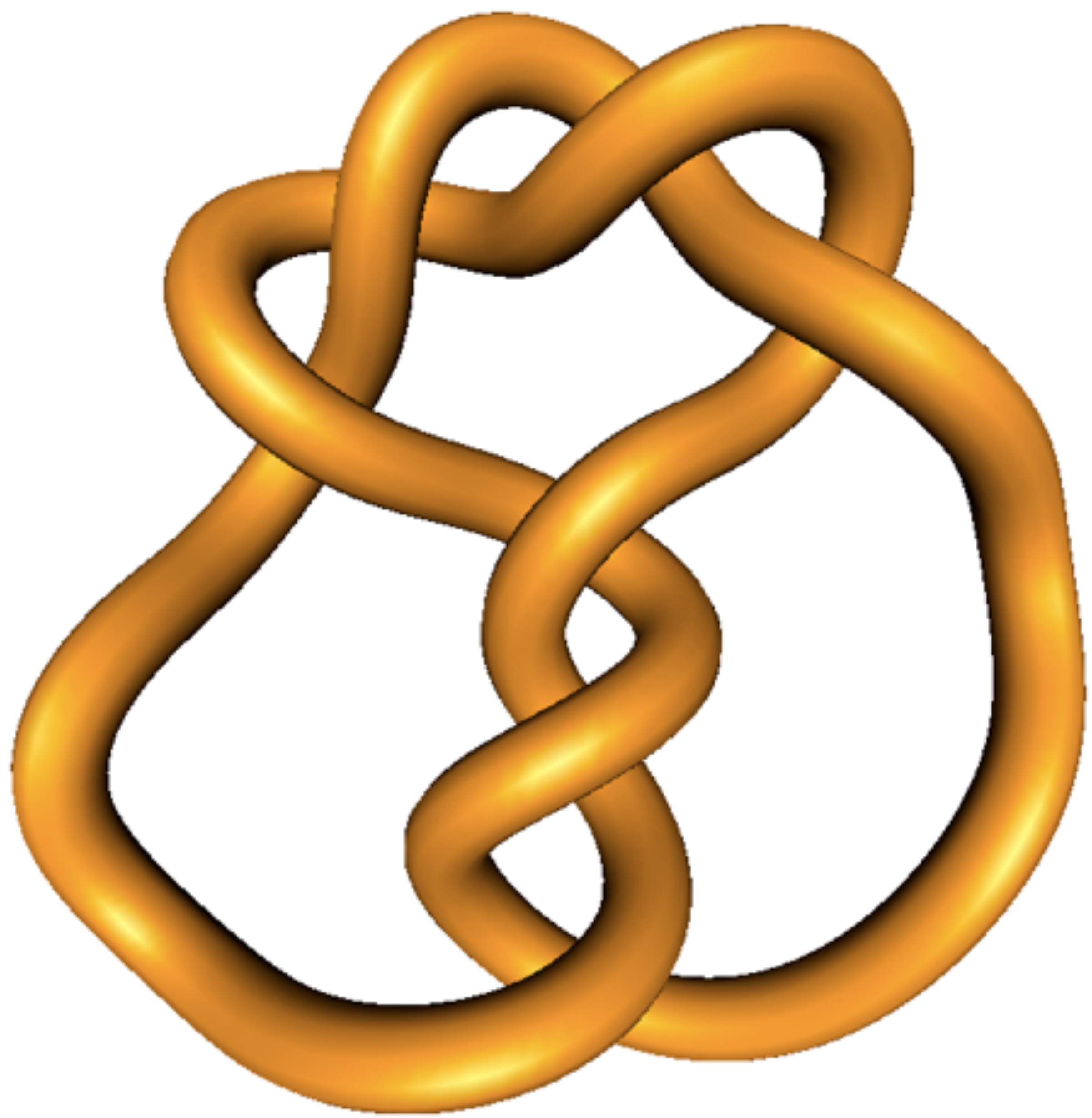
Applying the Borsuk-Ulam theorem:

- Notice that the surface of the Earth is S^2
- let the function f be temperature

- let the function g be wind speed
- **Conclusion:** at any moment, there are two points are opposite ends of the Earth with the exact same temperature and wind speed!

Knot theory

A mathematical knot is an embedded closed loop in three dimensional Euclidean space



Knot theory

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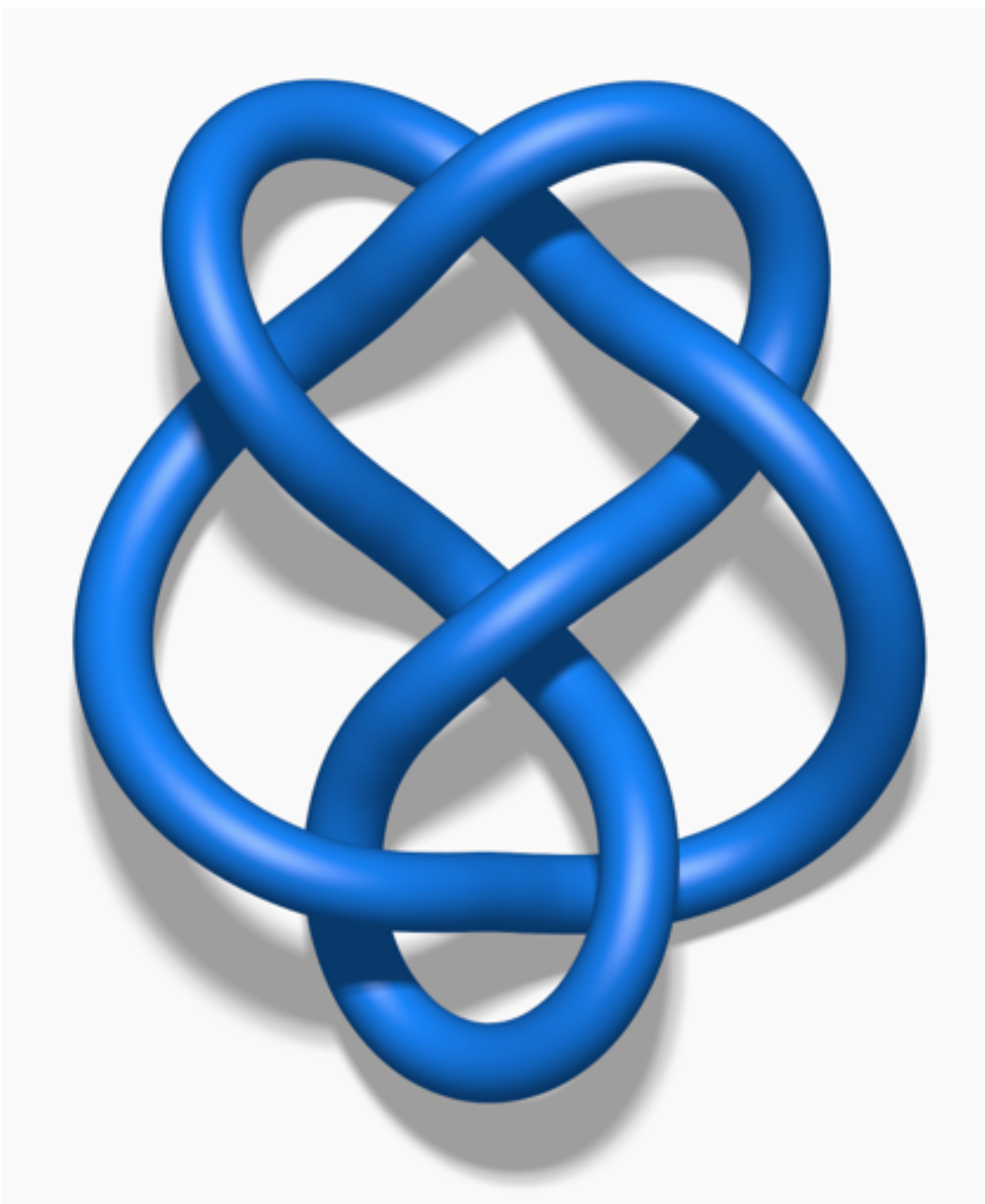
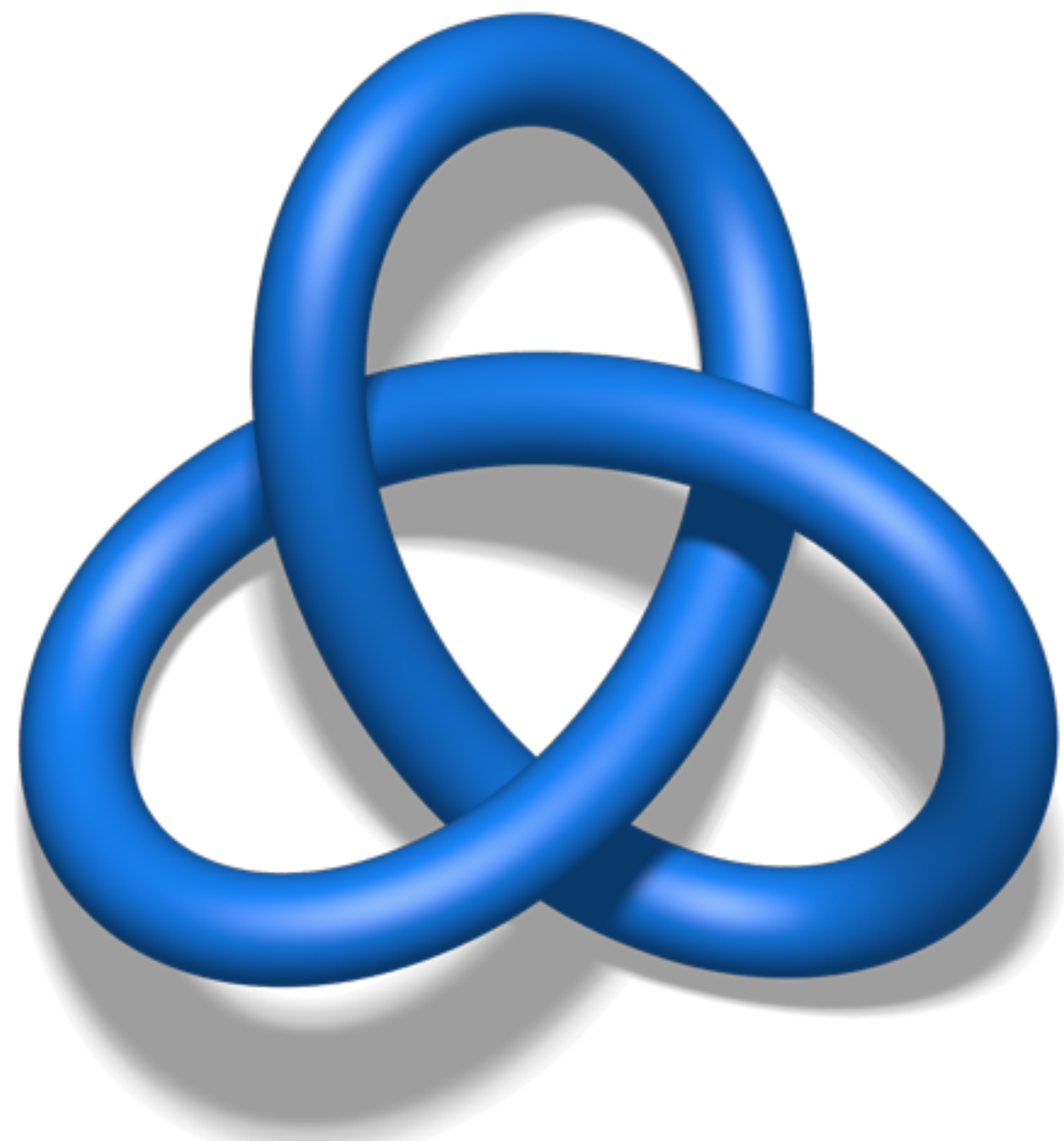
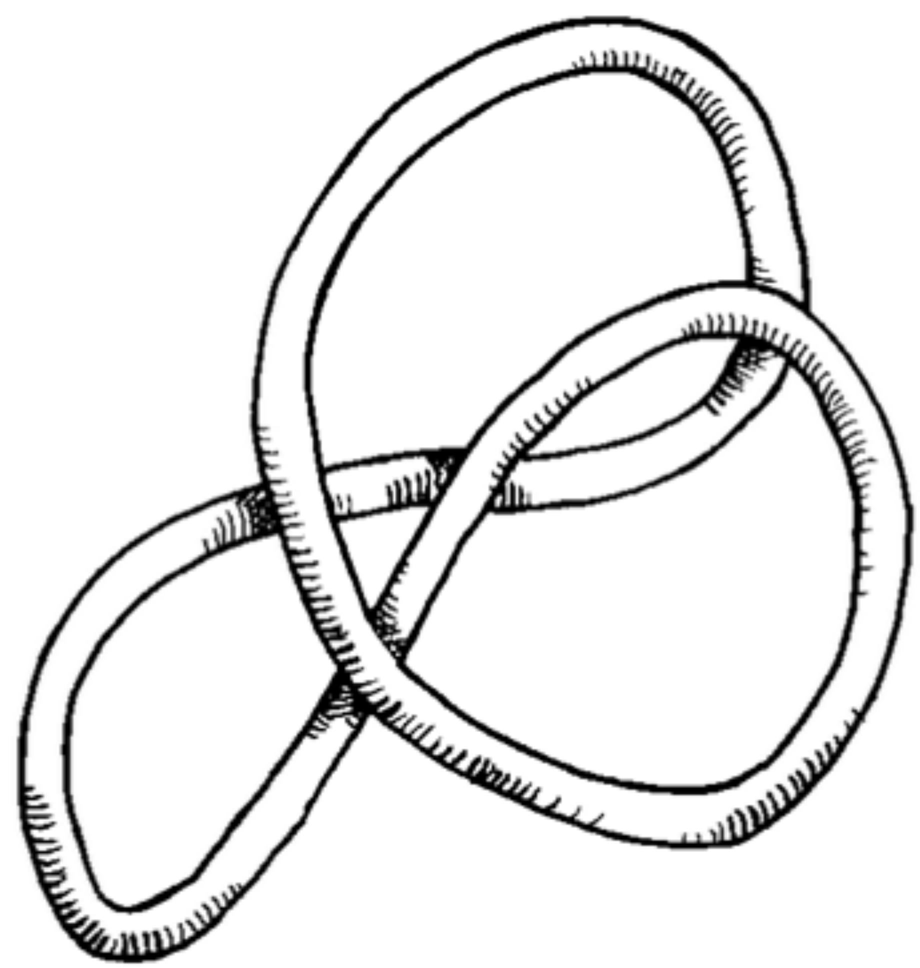




The “unknot”

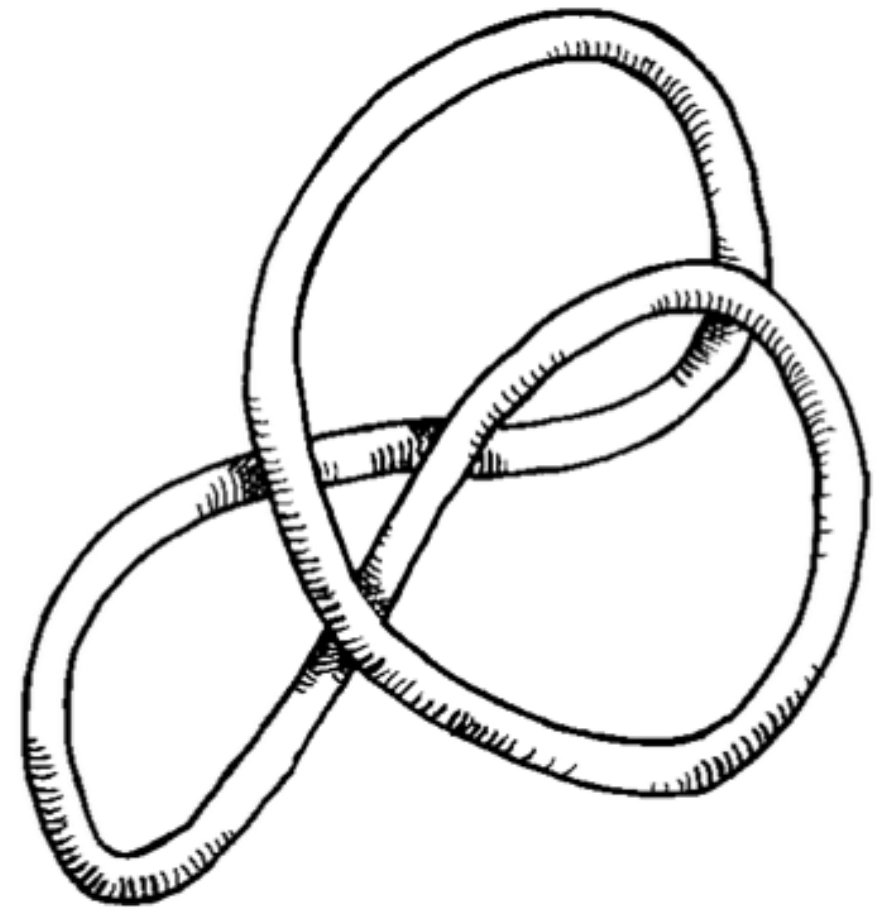
A knot which is not really “knotted” at all is called the unknot

A knot can be deformed arbitrarily as long as it never passes through itself!

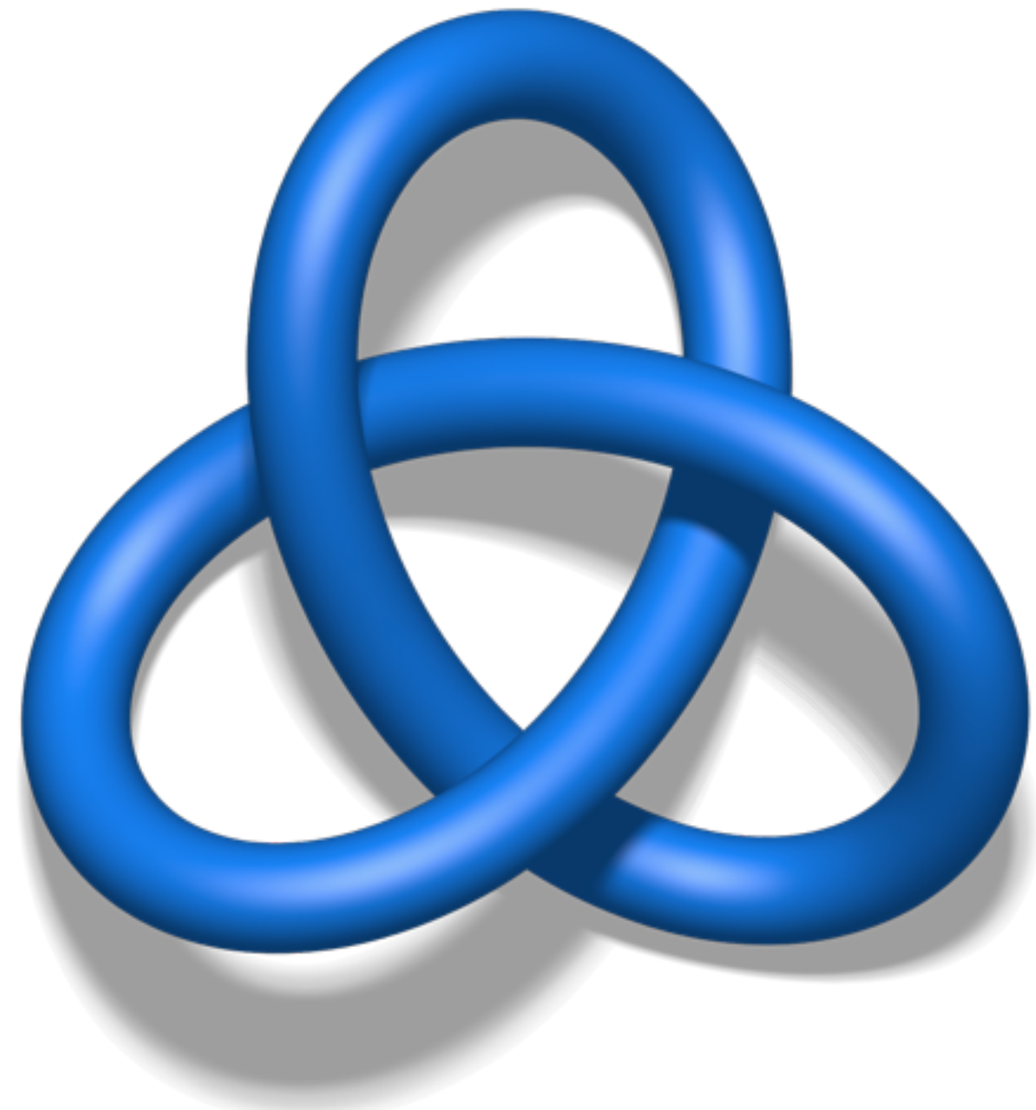


unknot!

$$\simeq$$



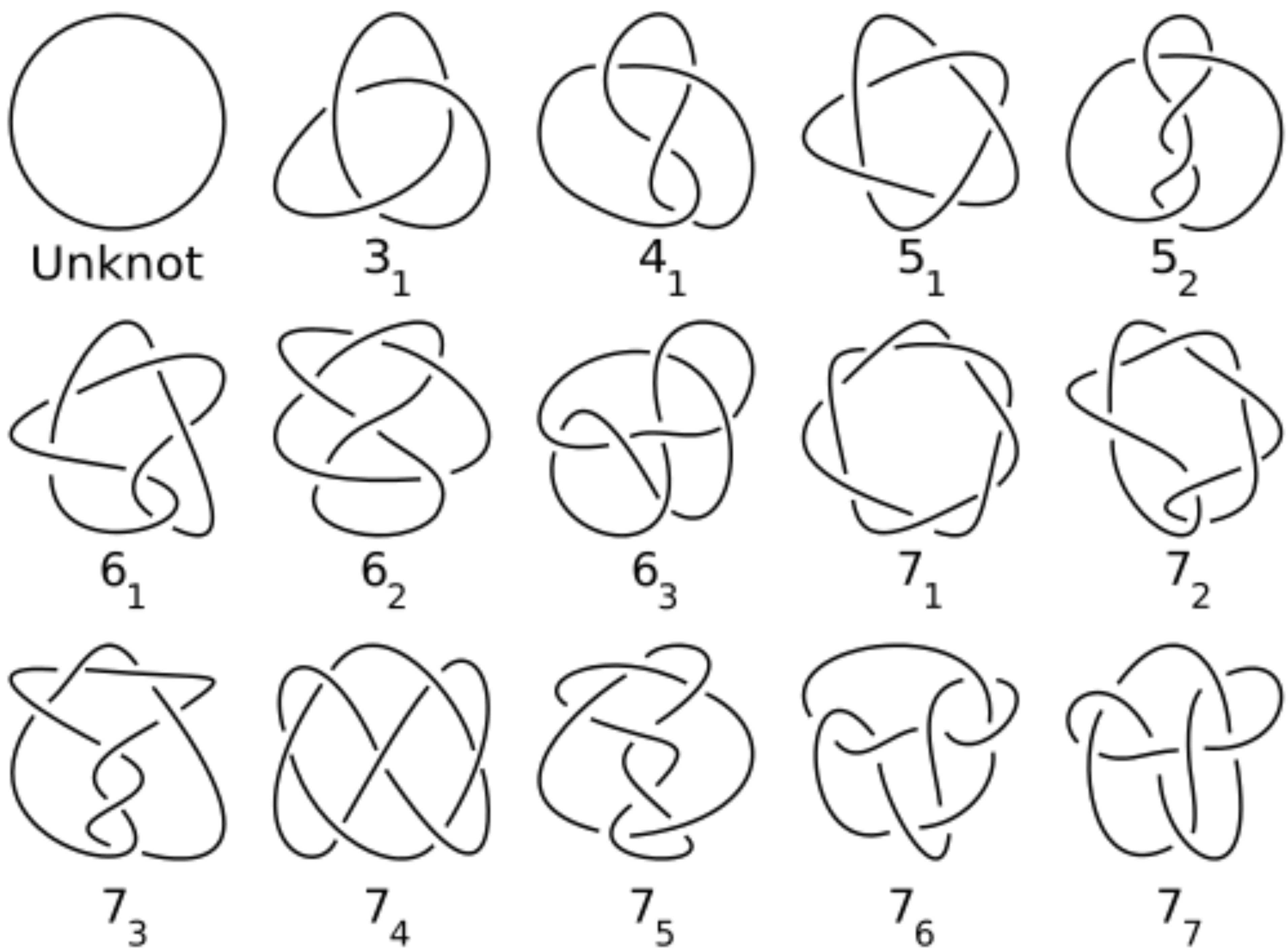
$\neq$



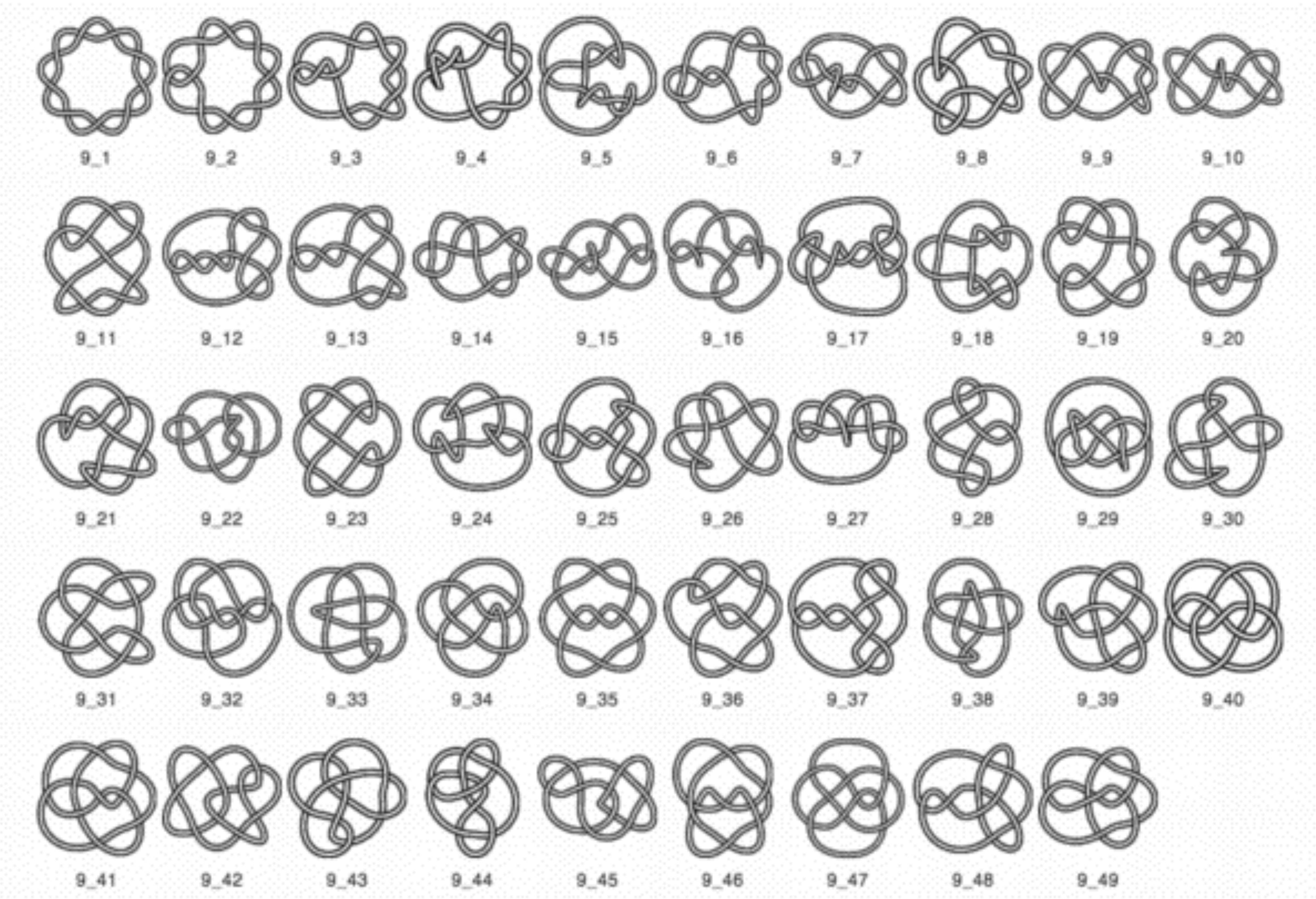
unknot

“trefoil”

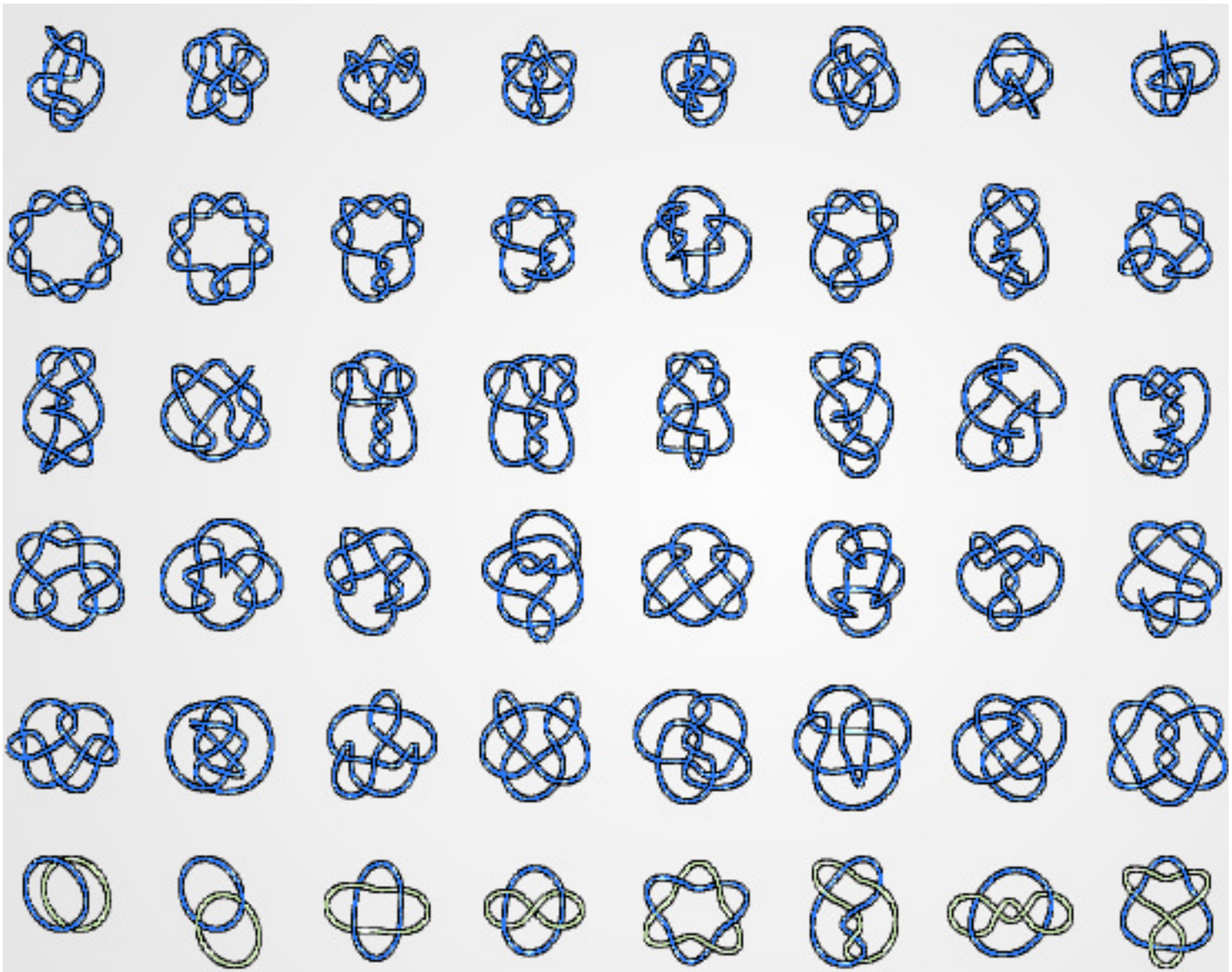
There are lots of complicated inequivalent knots



There are lots of complicated inequivalent knots



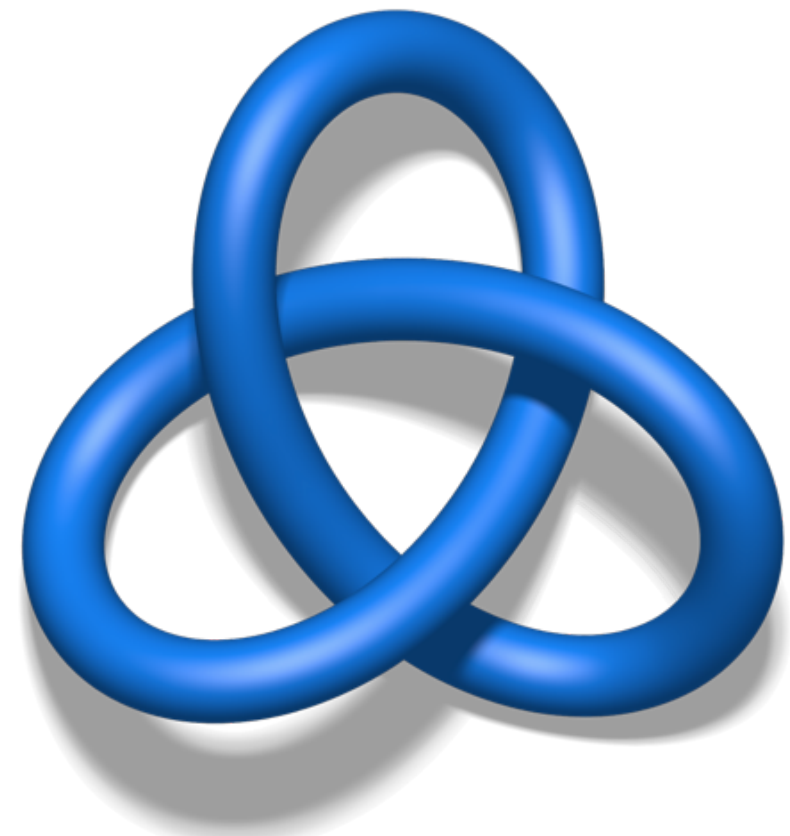
There are lots of complicated inequivalent knots



Baby example of a topological problem: PROVE that the unknot and the trefoil are inequivalent knots!

unknot

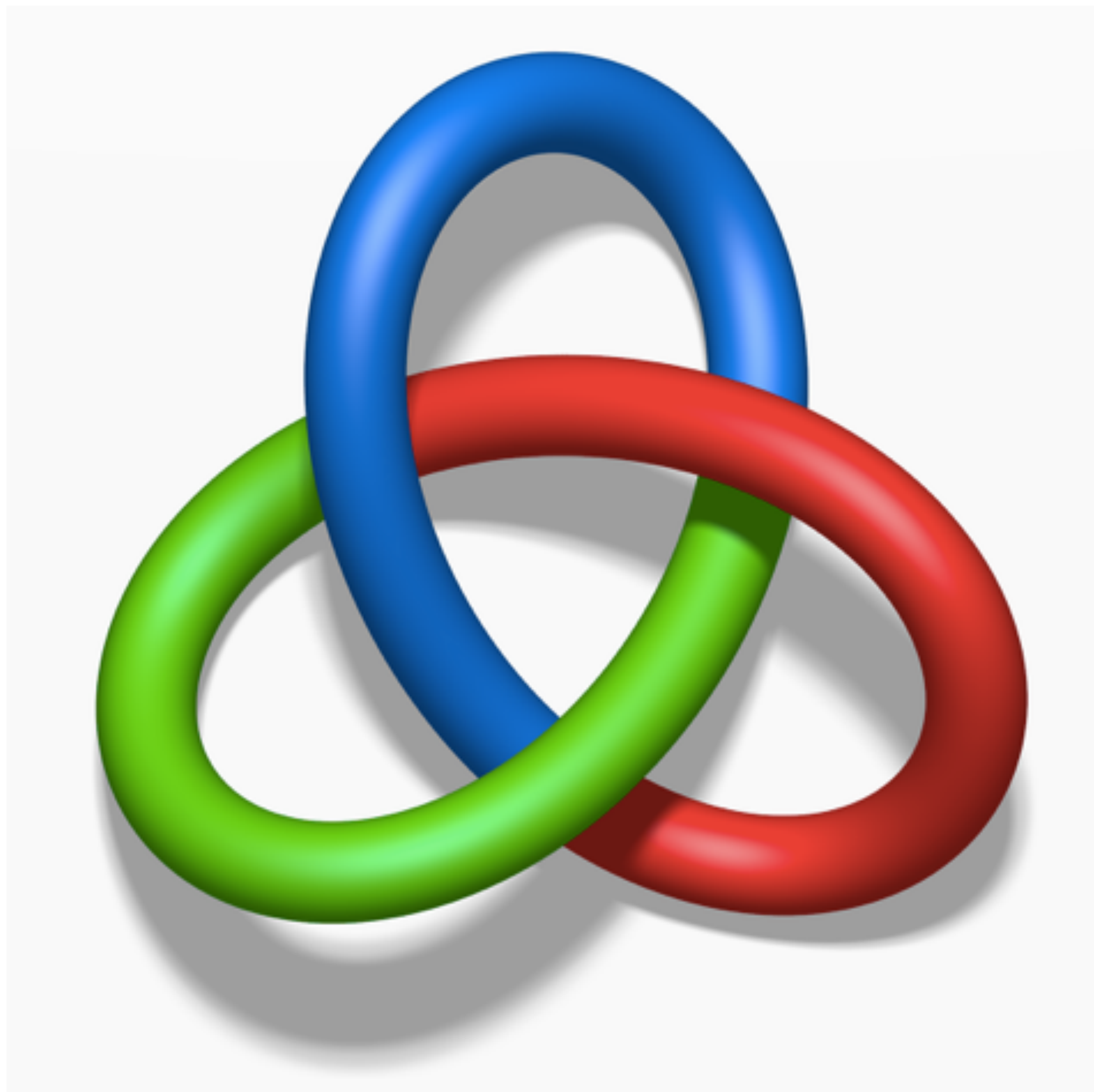
$\neq$



“trefoil”

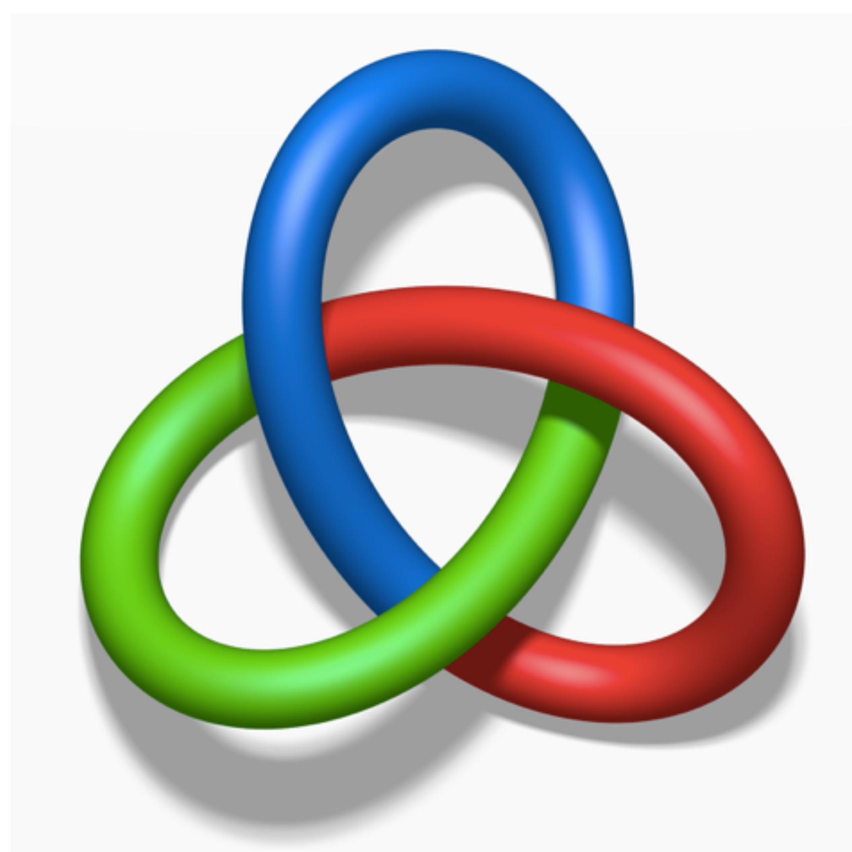
How can a mathematician *prove* that there isn't some super complicated sequence of moves that turns one into the other???

The simplest way to prove the trefoil is knotted: tricolorability

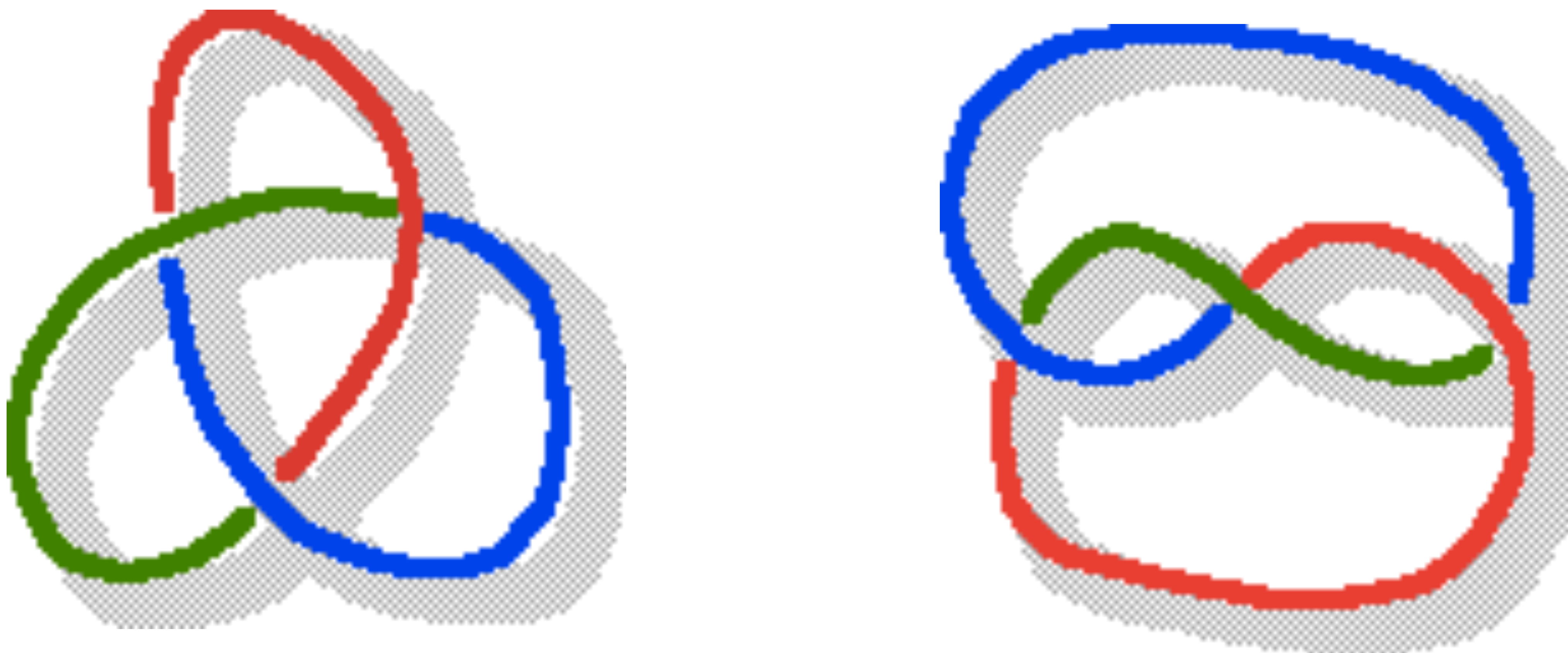


A knot is “tricolorable” if one can color it with three different colors, subject to the rules:

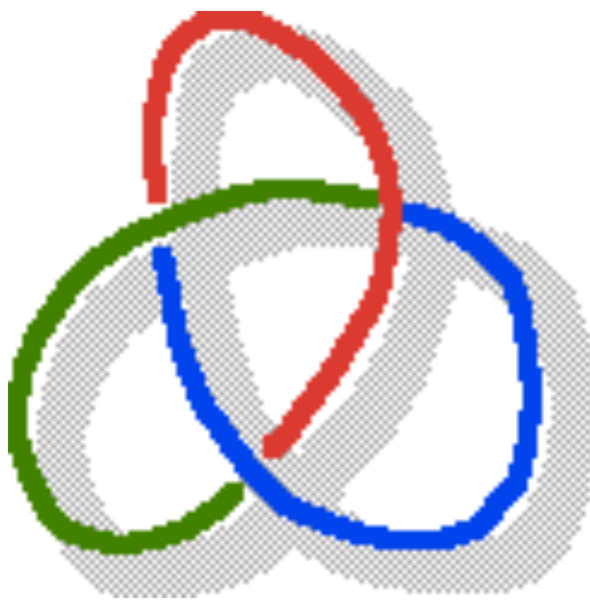
1. At least two colors must be used.
2. At each crossing, the three incident strands are either all the same color or all different colors.

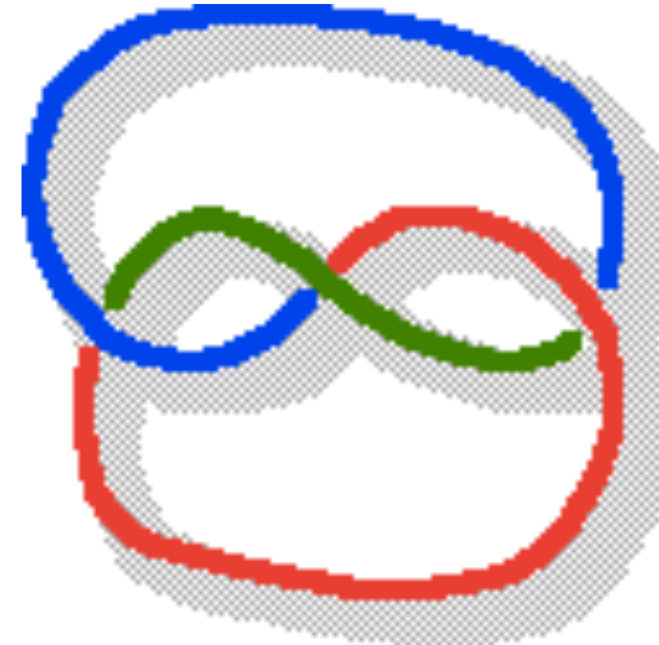


Tricolorability is an “invariant” of a knot



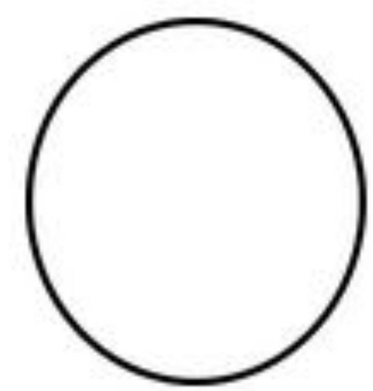
two examples of tricolored trefoil knots





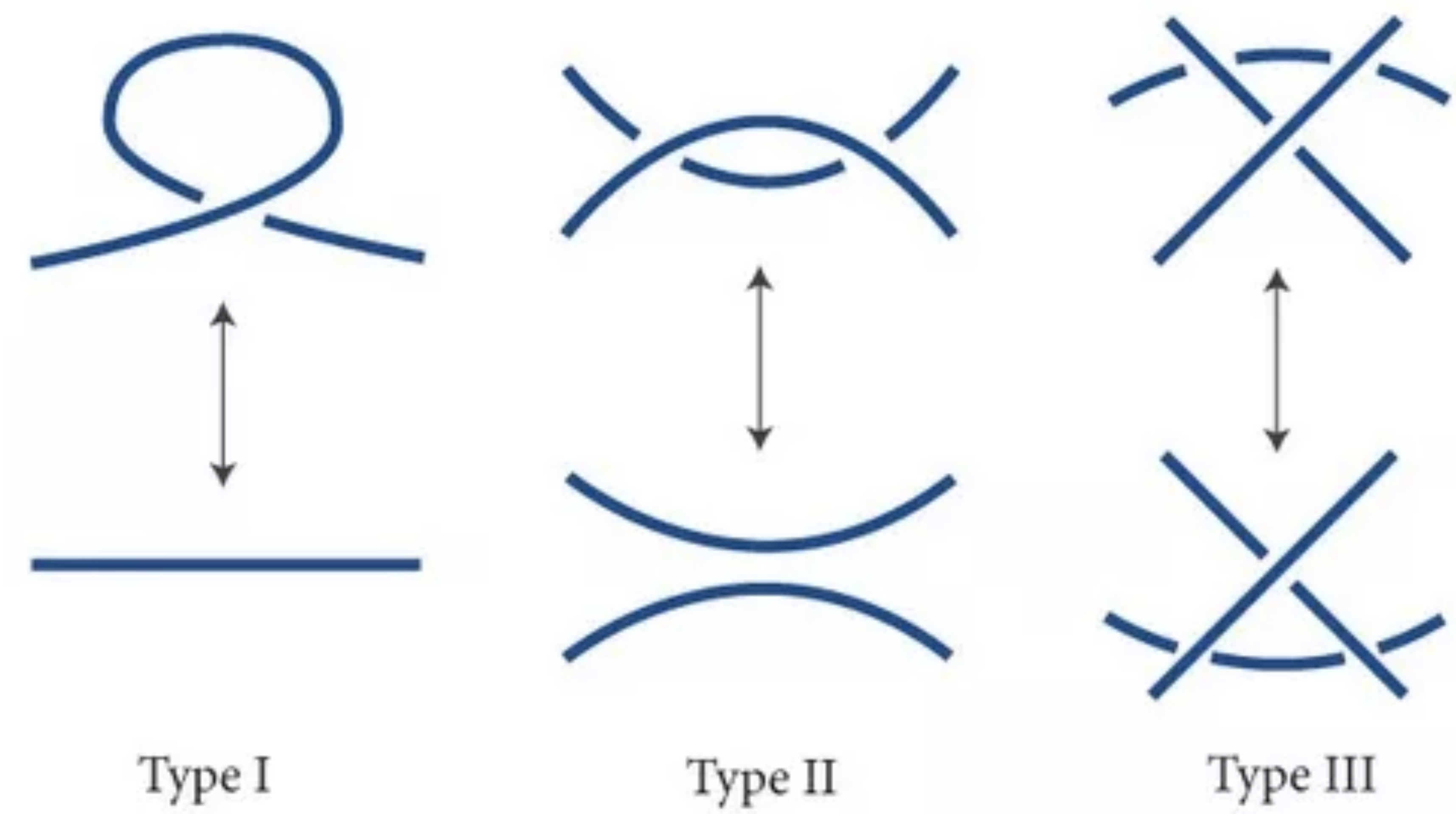
two examples of tricolored trefoil knots

=



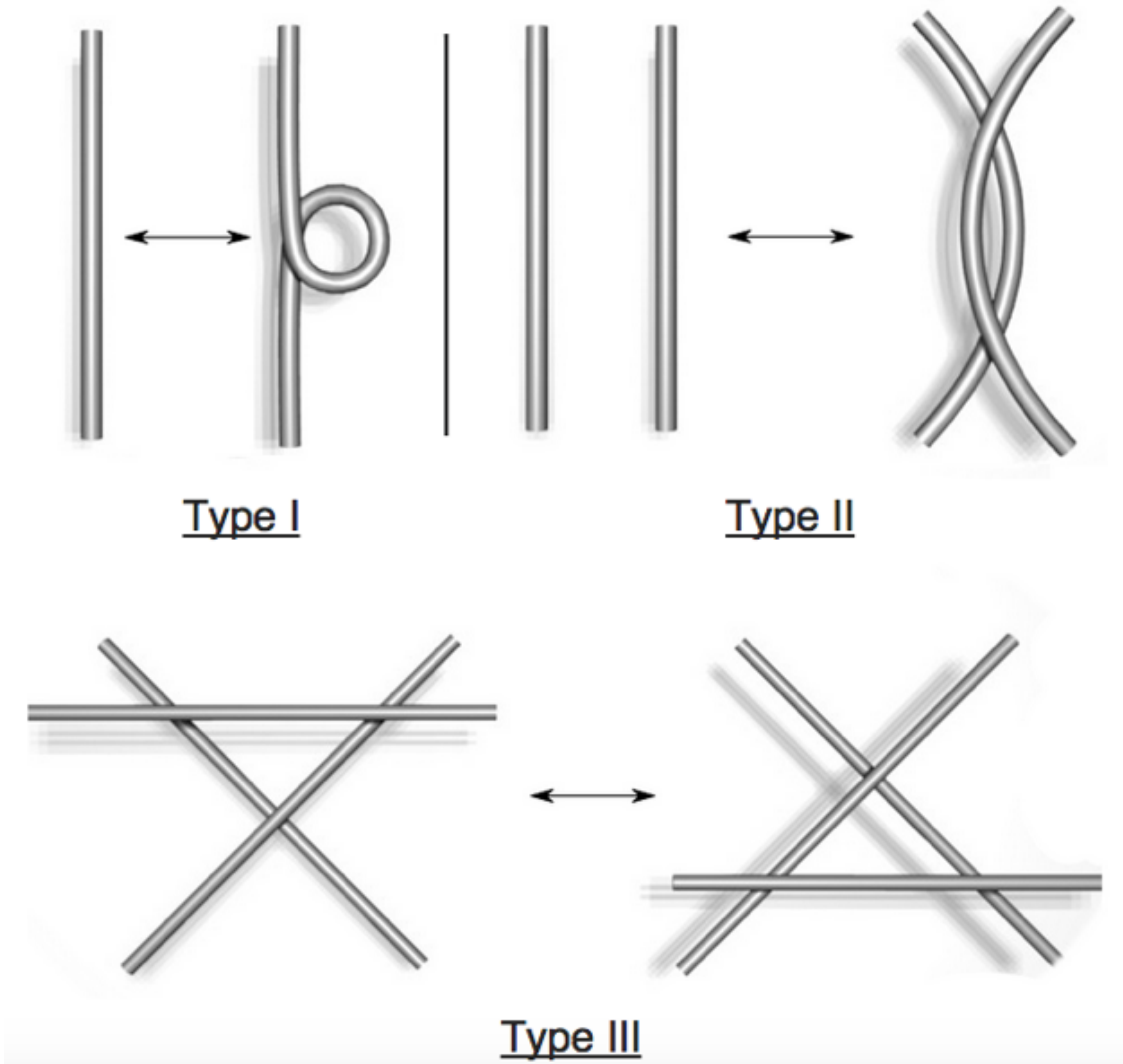
the unknot is *not* tricolorable!

How to prove the tricolorability is a knot invariant?

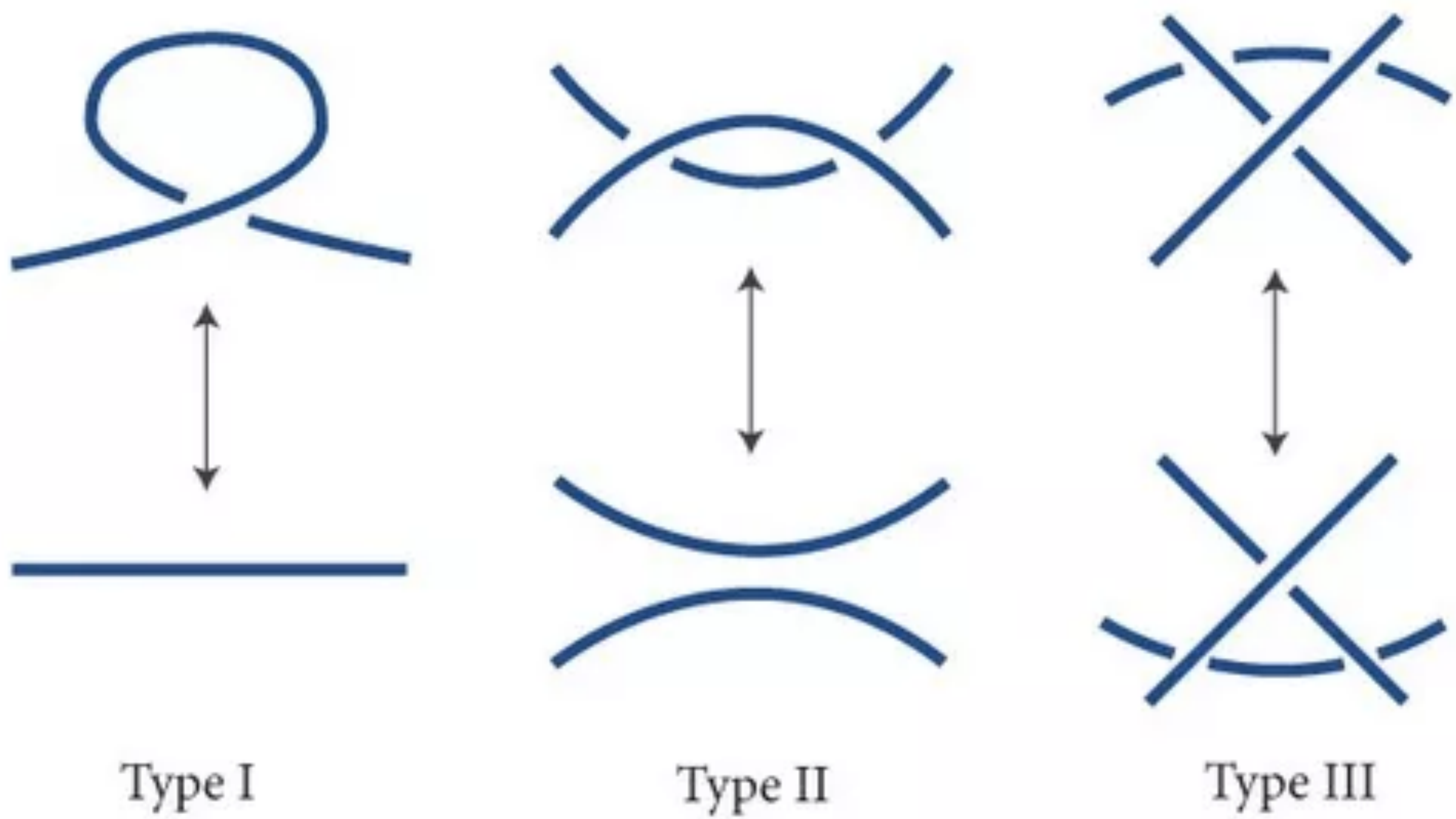


Reidemeister moves!

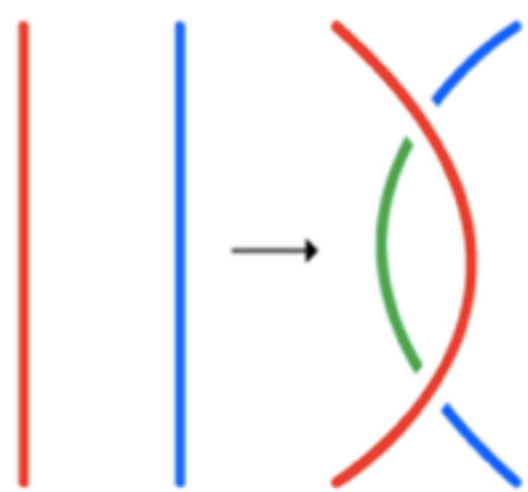
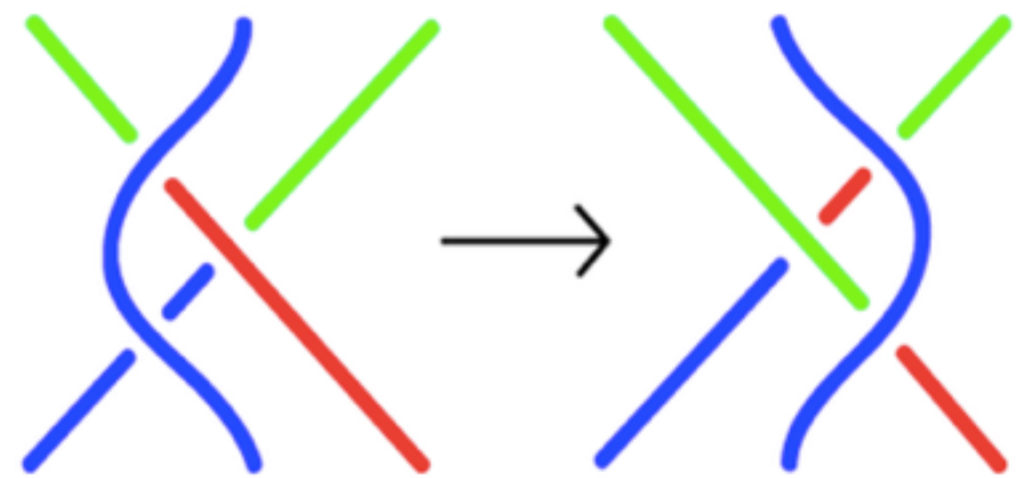
The three Reidemeister moves

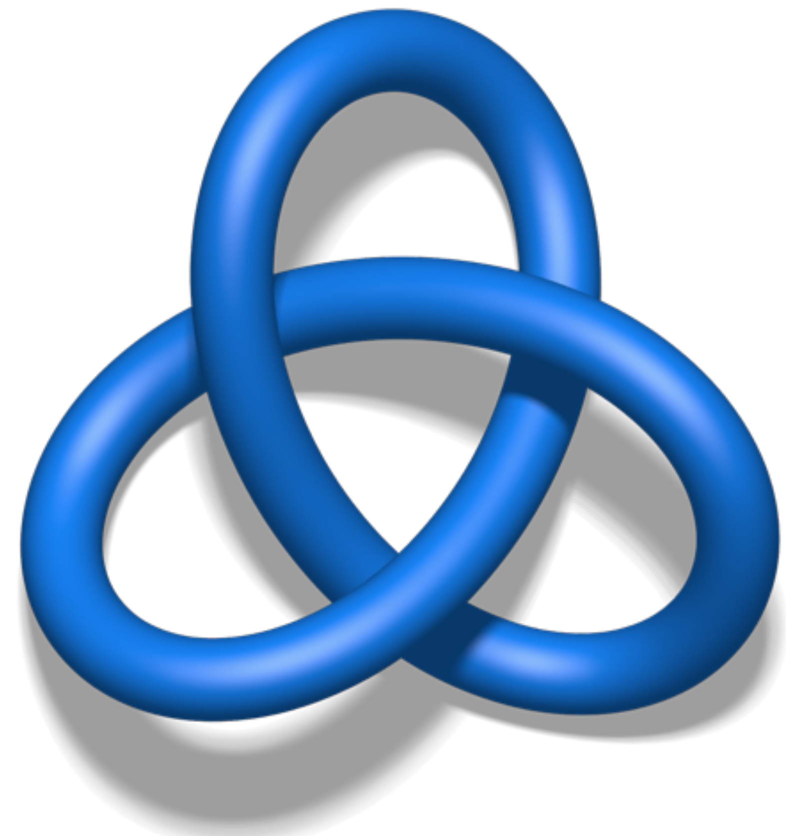


Any two equivalent knots must differ by a (possibly very long) sequence of Reidemeister moves



Not too hard to show that tricolorability is preserved under Reidemeister moves

Reidemeister Move I is tricolorable.	Reidemeister Move II is tricolorable.	Reidemeister Move III is tricolorable.
		

 $\neq$  !

Nowadays mathematicians have a powerful arsenal of knot invariants capable of distinguishing very complicated knots

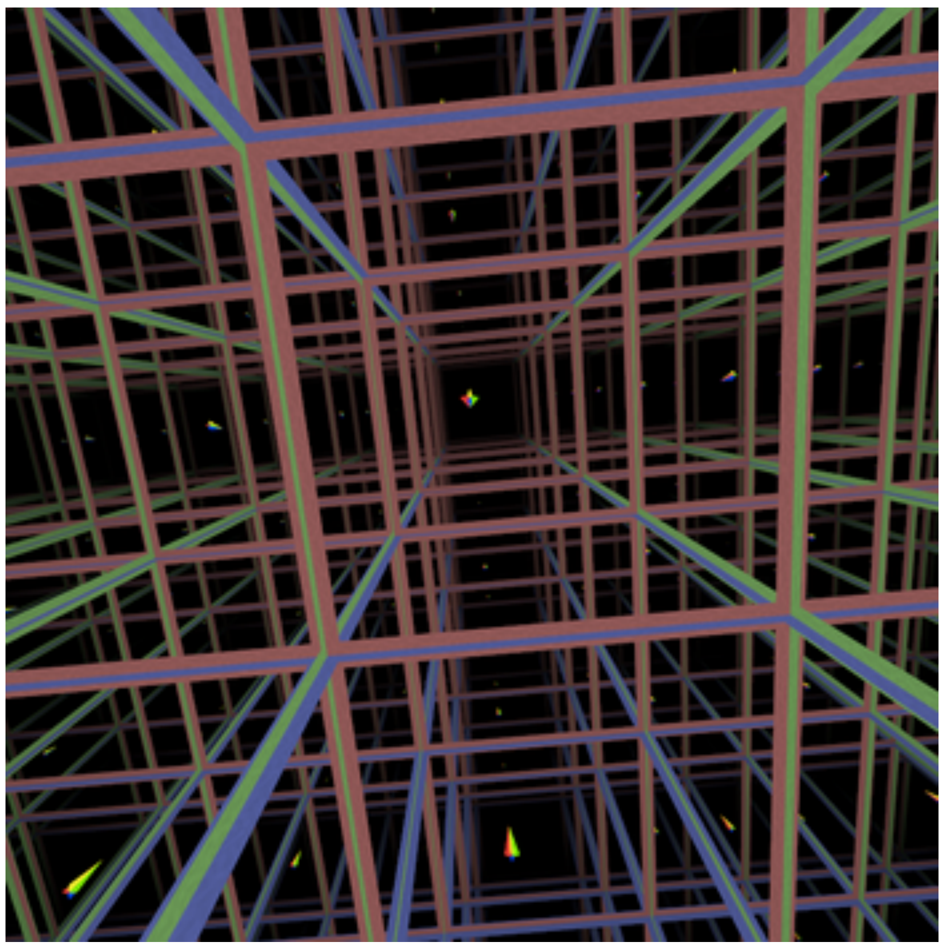
- Knot fundamental groups
- Knot polynomials
- Knot Floer homology
- Quantum knot invariants
- Khovanov homology

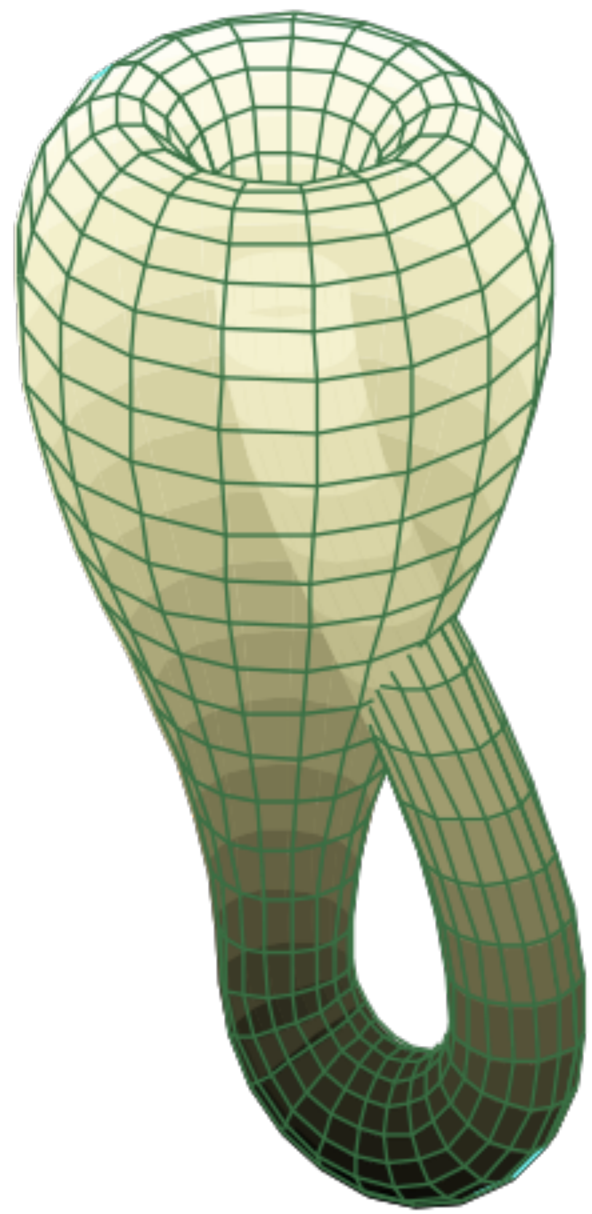
Big open question: is there a knot invariant which can distinguish all inequivalent knots?

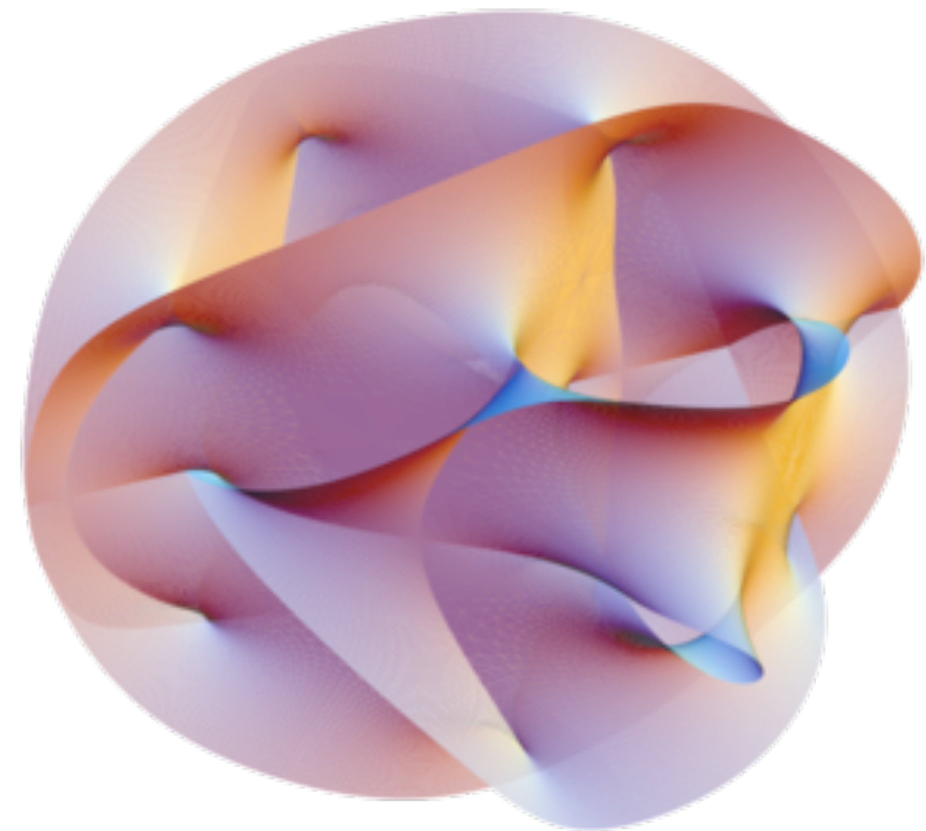
10_{161}

10_{162}

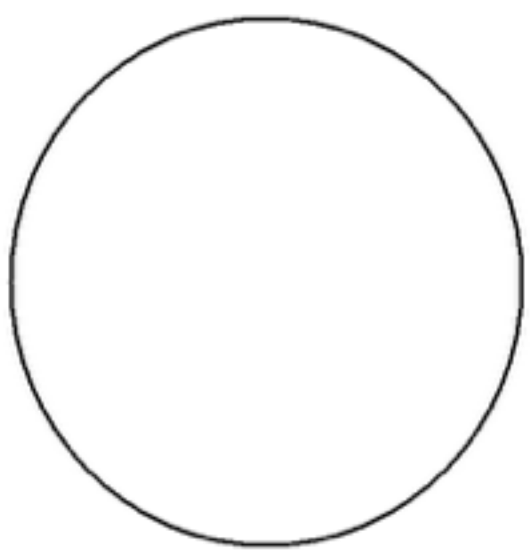
Manifolds







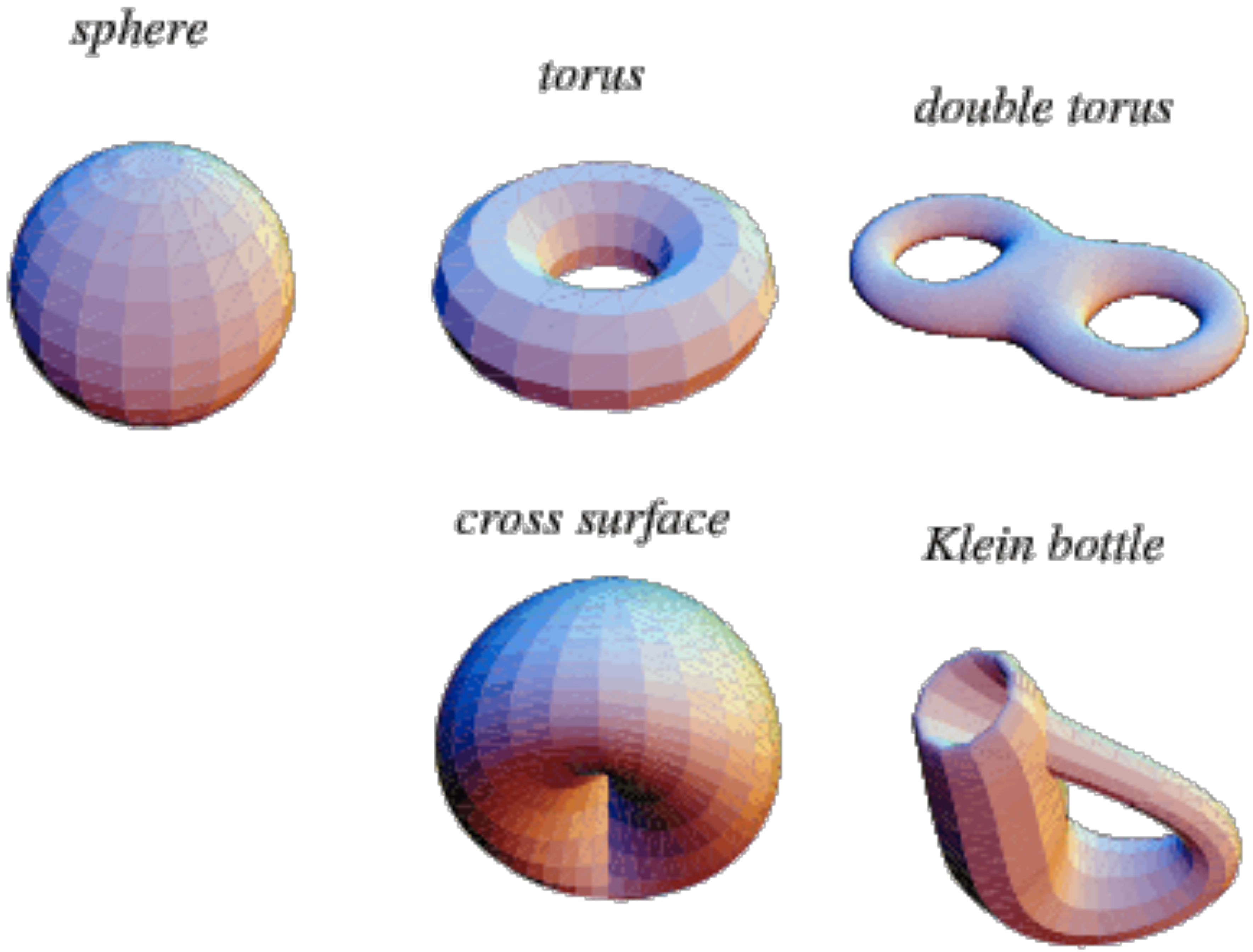
- A *manifold* is a space which looks like ordinary flat Euclidean space *locally*, but globally may be quite different
- These exist in every dimension, but are extremely hard to visualize in dimensions greater than two



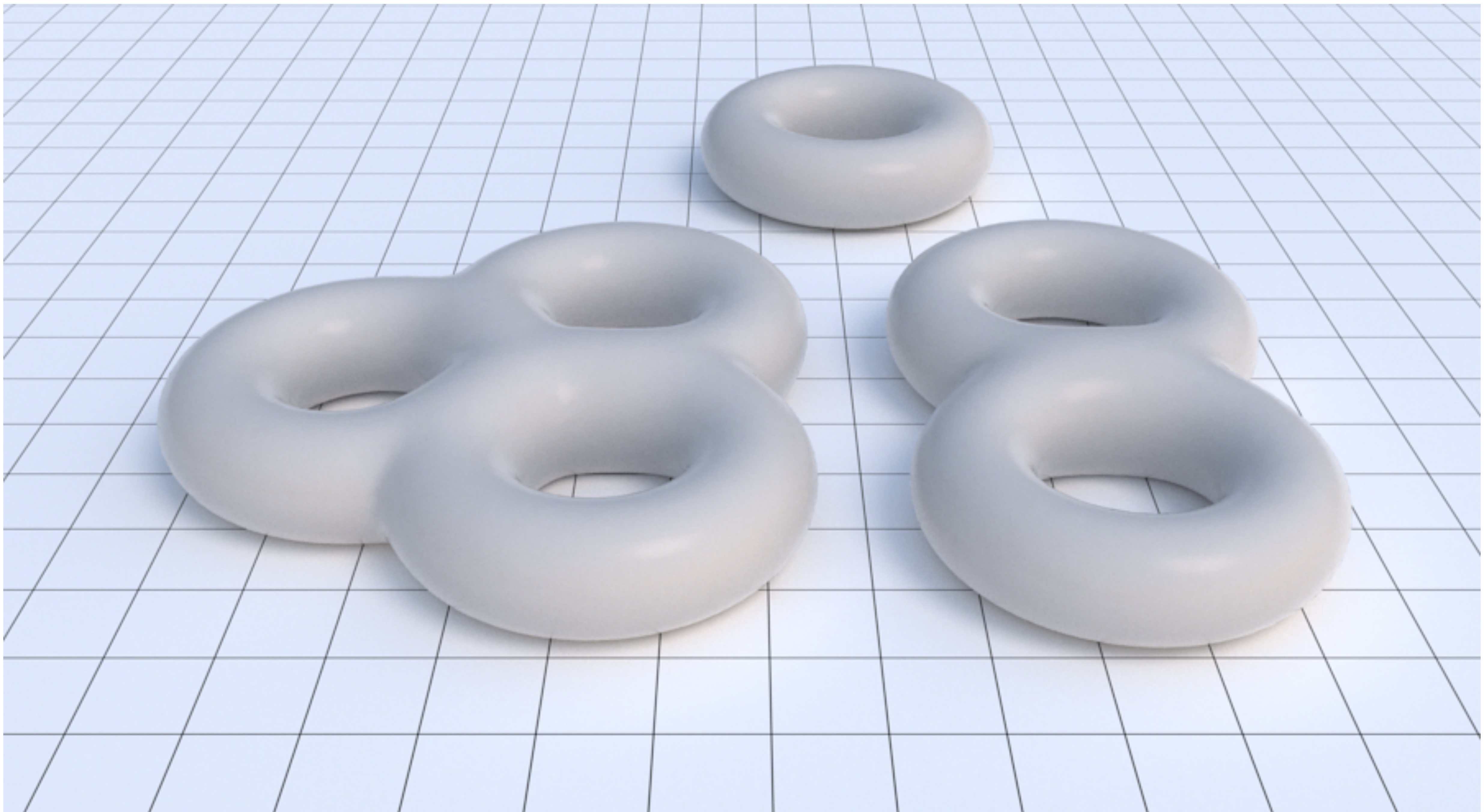
the circle is a
1-dimensional manifold

the figure eight is
not a manifold

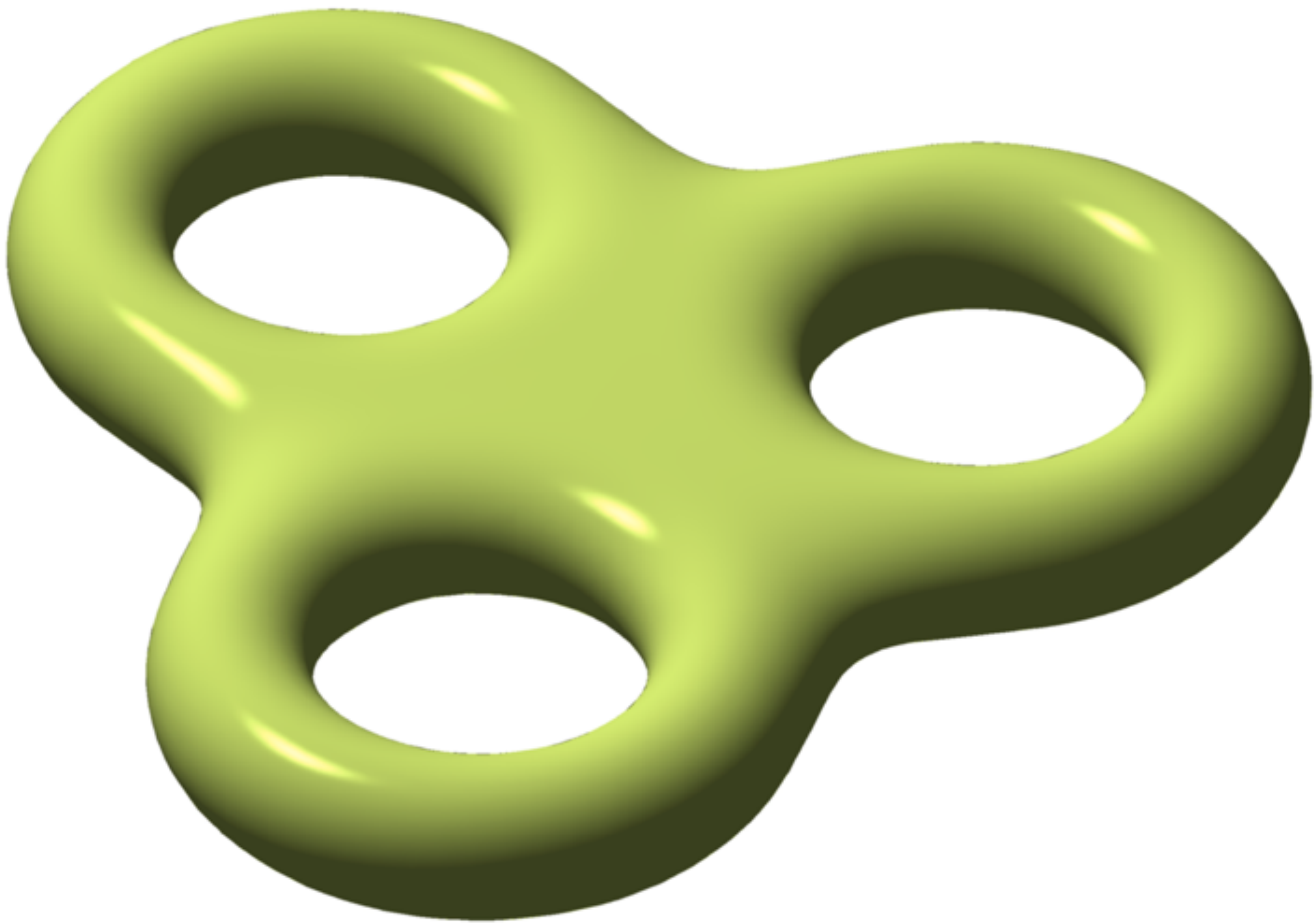
Classification of 2-dimensional manifolds (a.k.a. *surfaces*)



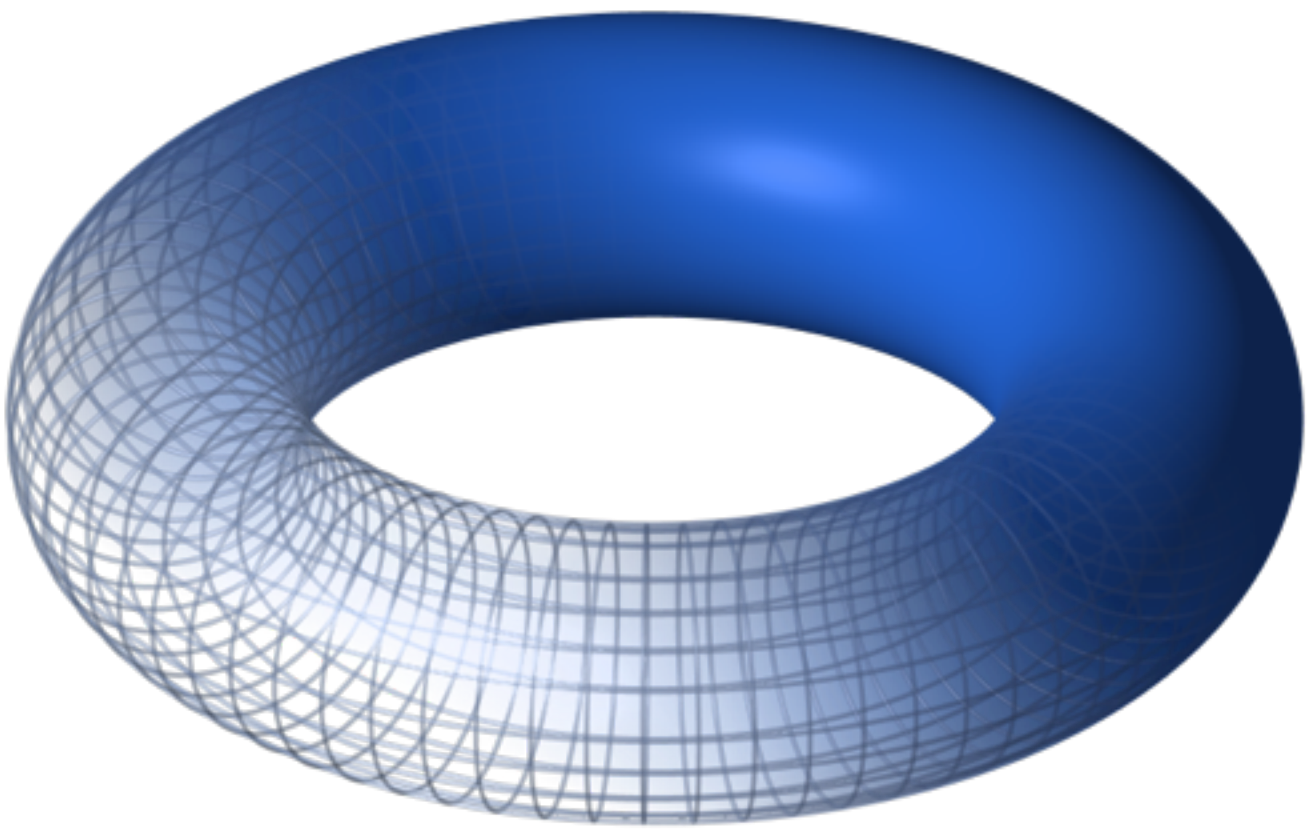
The “genus” (number of holes) can be arbitrarily large



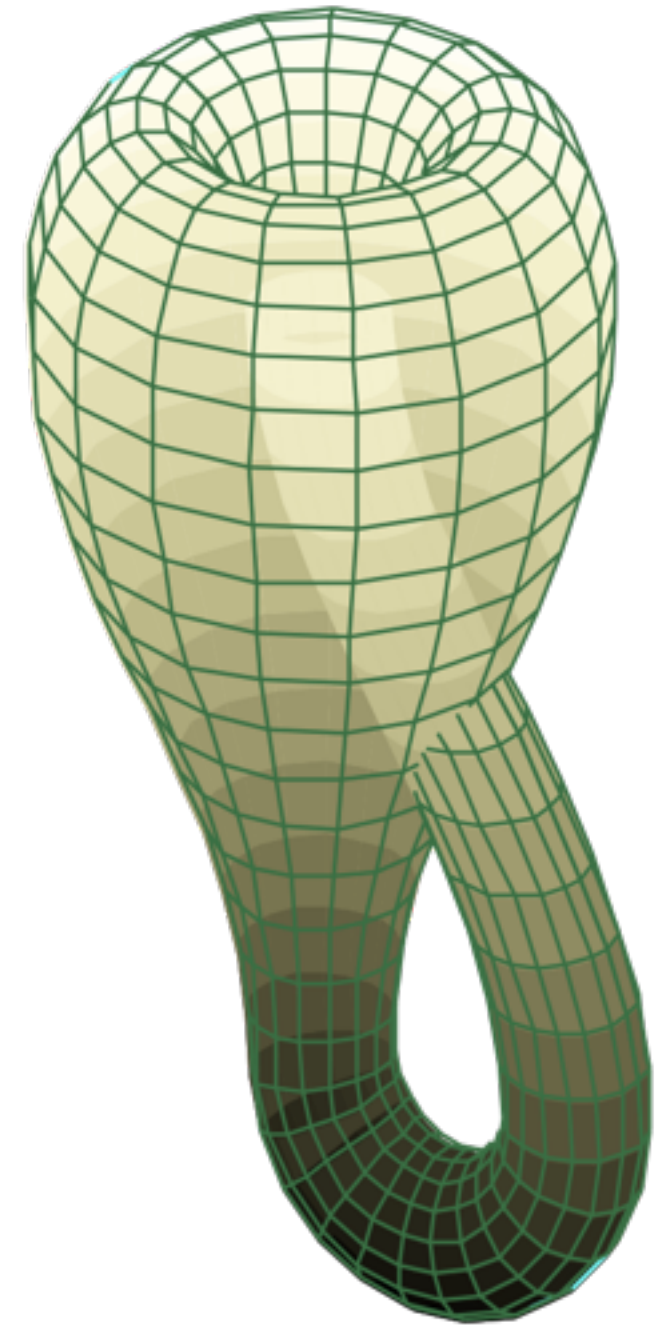
“triple torus”, a.k.a. the genus 3 surface



There are “orientable” and “nonorientable” versions of surfaces



torus

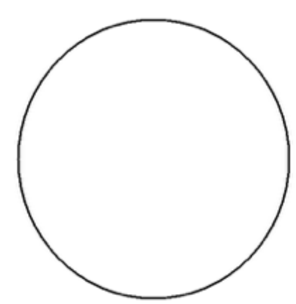


Klein bottle

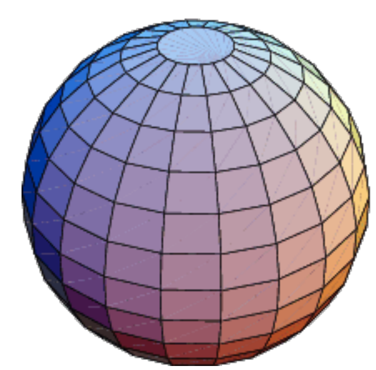
Three-dimensional manifolds:

- three dimensional flat Euclidean space $\mathbb{R}^3$
- the three-sphere $S^3 = \{x^2 + y^2 + z^2 + w^2 = 1\}$
- the "product" of the circle and the sphere:

$S^1 \times S^2$ =



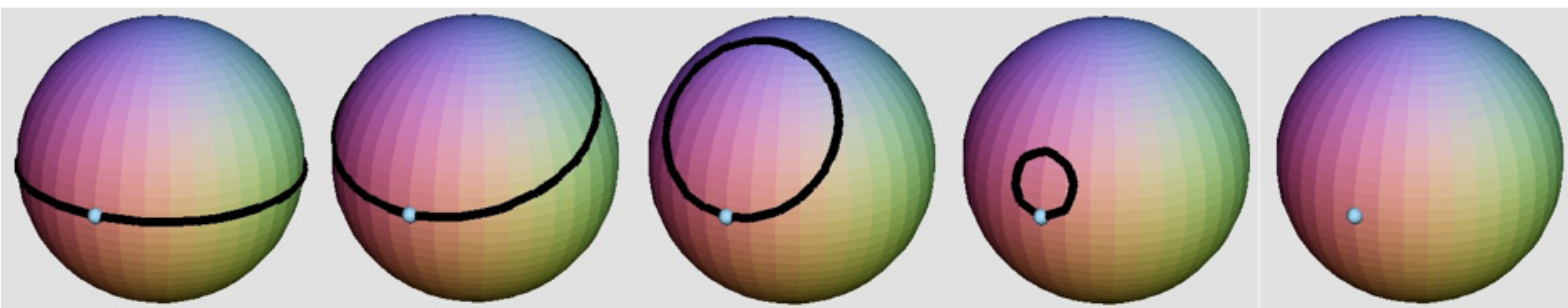
x



We are living in a three-manifold right now

Simply-connected spaces:

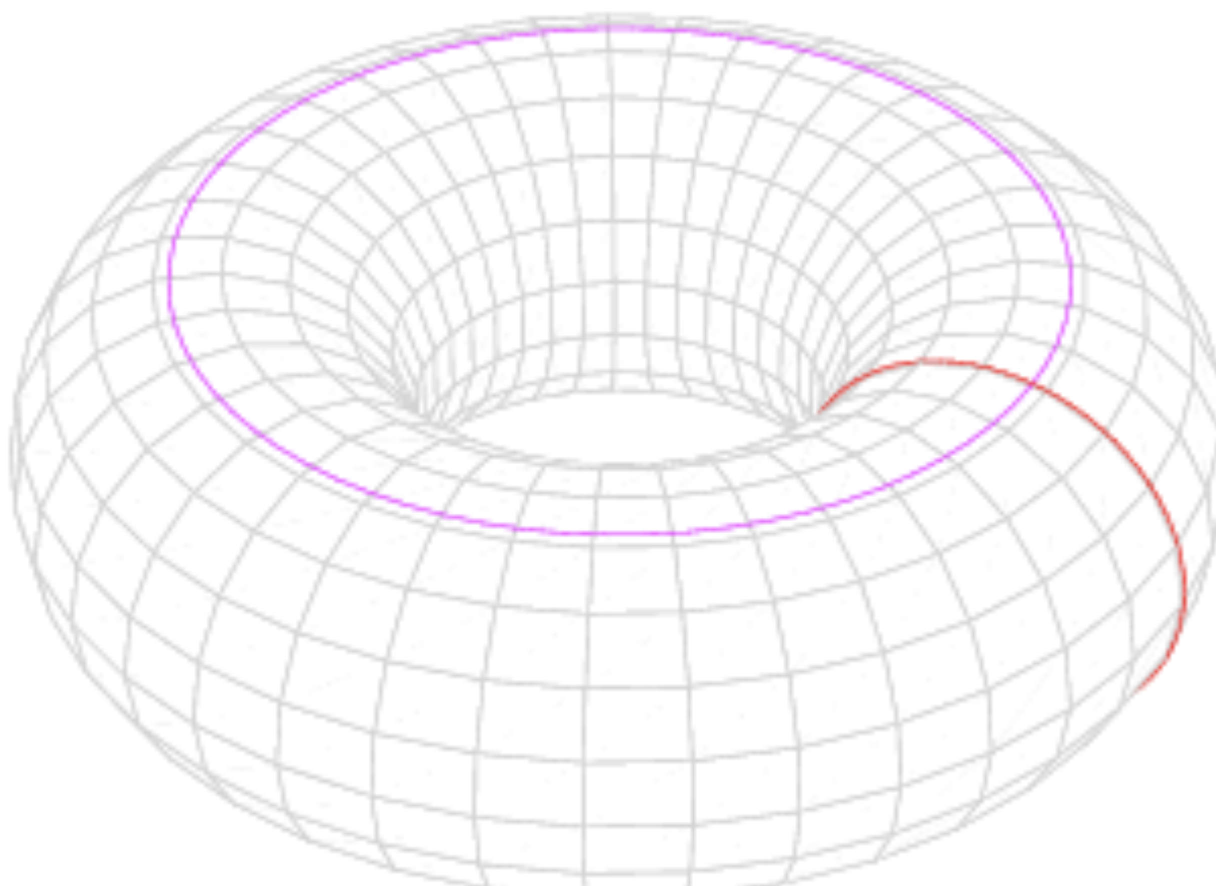
- A space is *simply-connected* if any loop can be continuously deformed into a point



the two-sphere is simply-connected

Simply-connected spaces:

- A space is *simply-connected* if any loop can be continuously deformed into a point



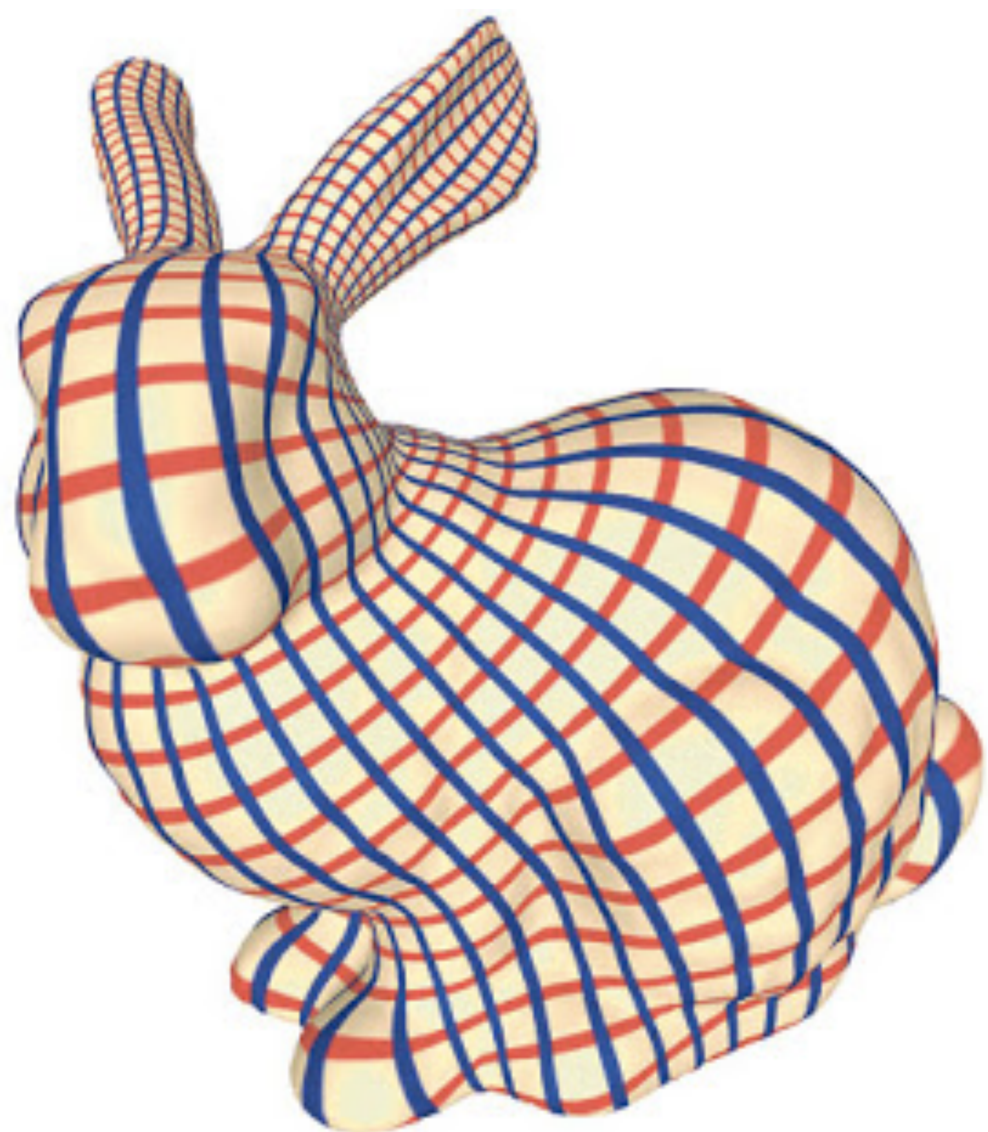
the torus is *not* simply-connected!

The Poincare Conjecture:

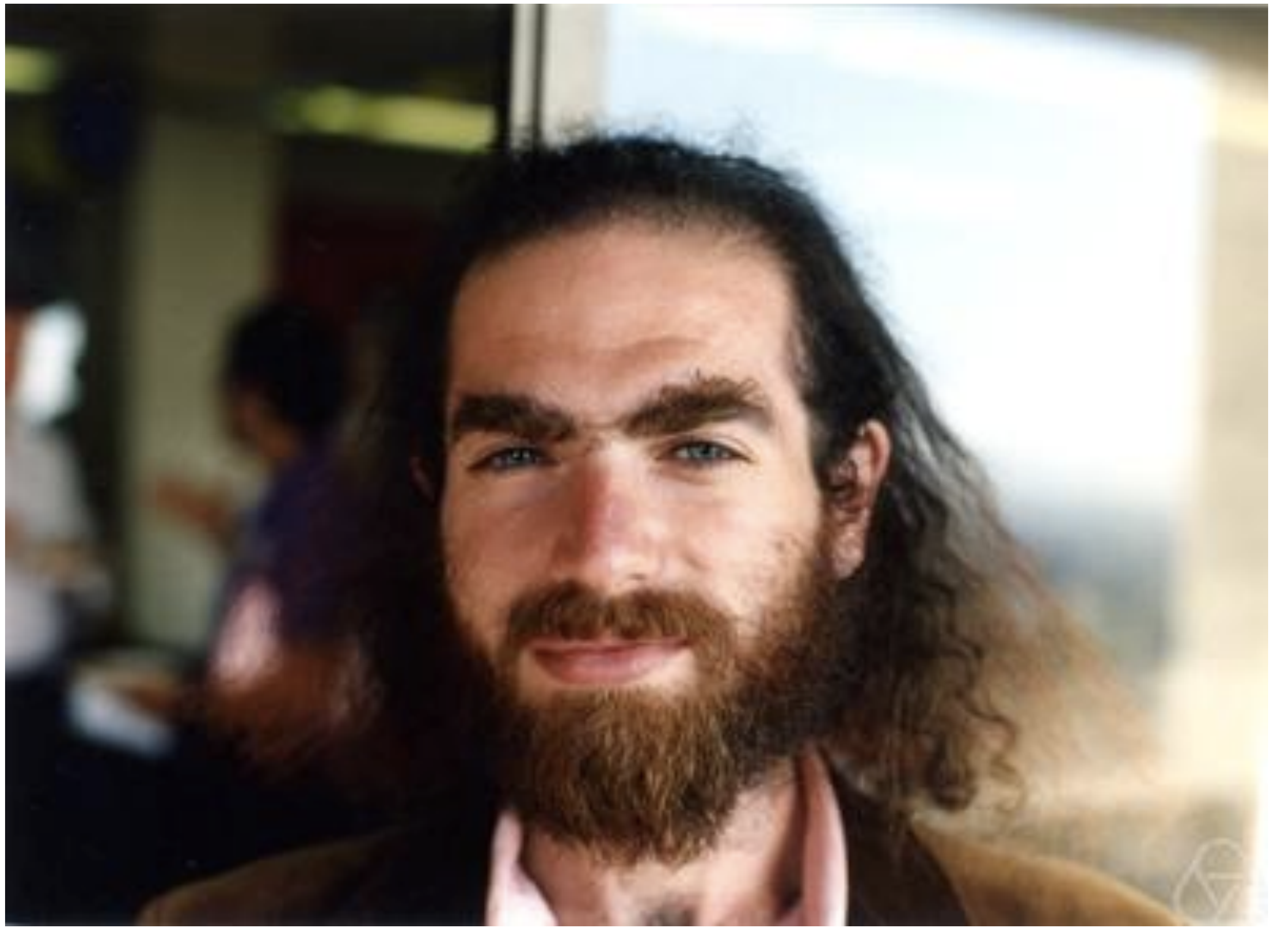
Henri Poincare, b. 1854

The Poincare Conjecture:

- Asks whether every simply-connected 3-manifold is equivalent to three-sphere

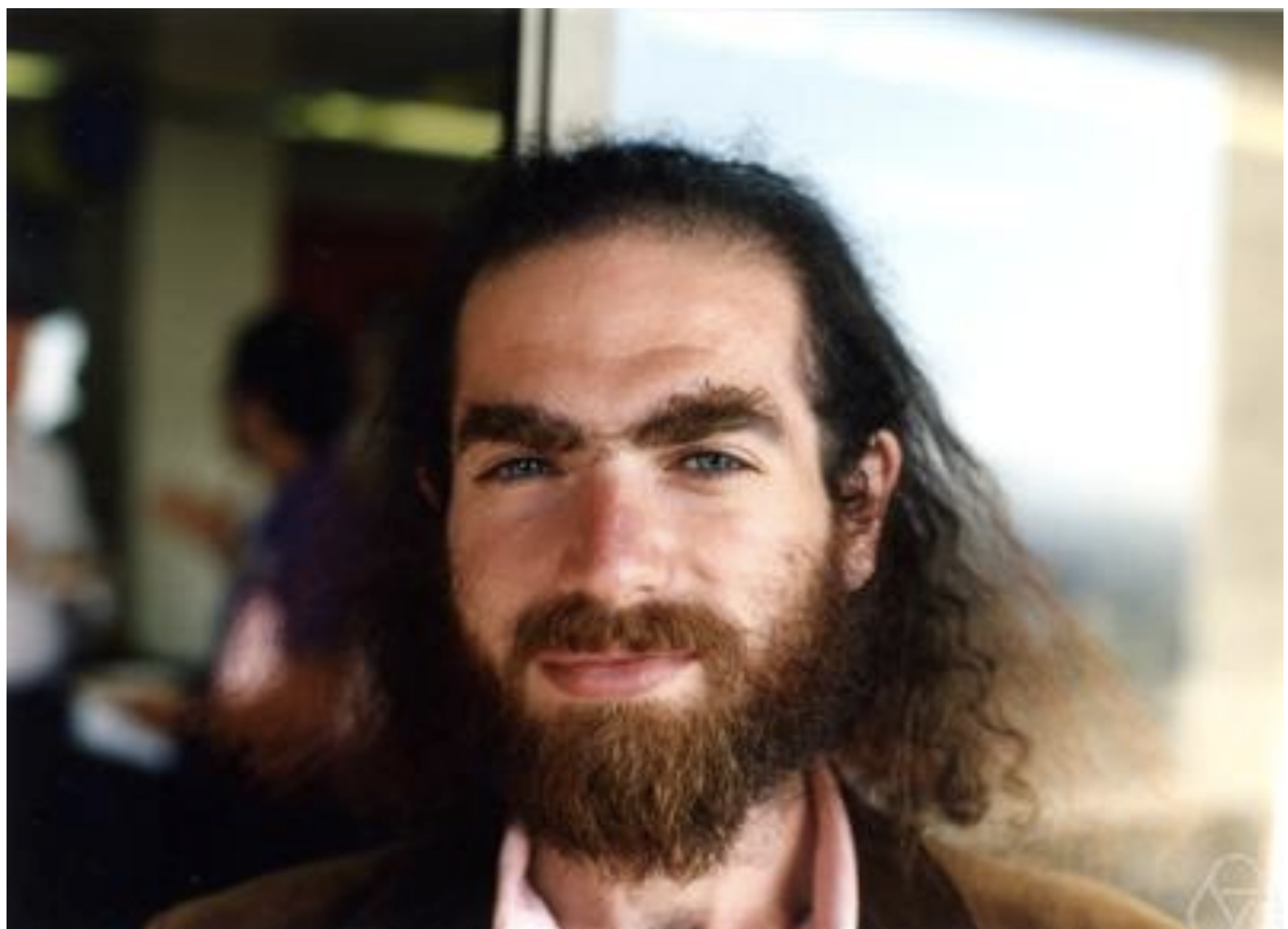


It was proved by Perelman in 2003



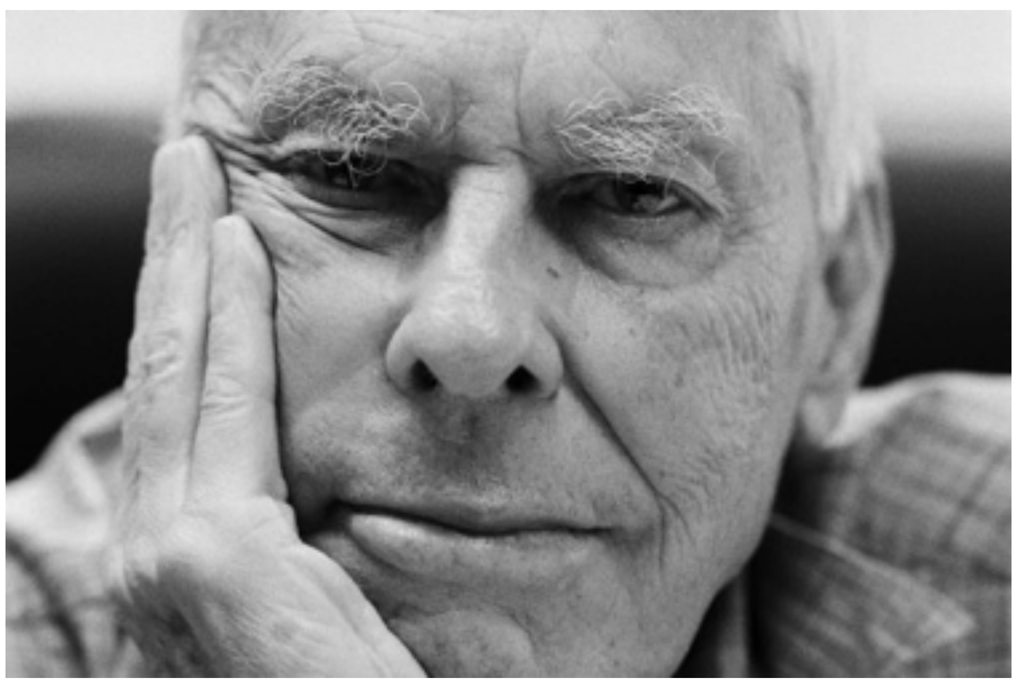
Grigori Perelman, b. 1966

- His prove utilized a new technique from geometric analysis called “Ricci flow”
- For his result he was offered both a Fields medal and a $1,000,000 Millenium Prize, and turned them both down!



There is also analogues of the Poincare Conjecture in dimensions other than three

- In dimensions five and higher, it was proved and disproved by Milnor and Smale

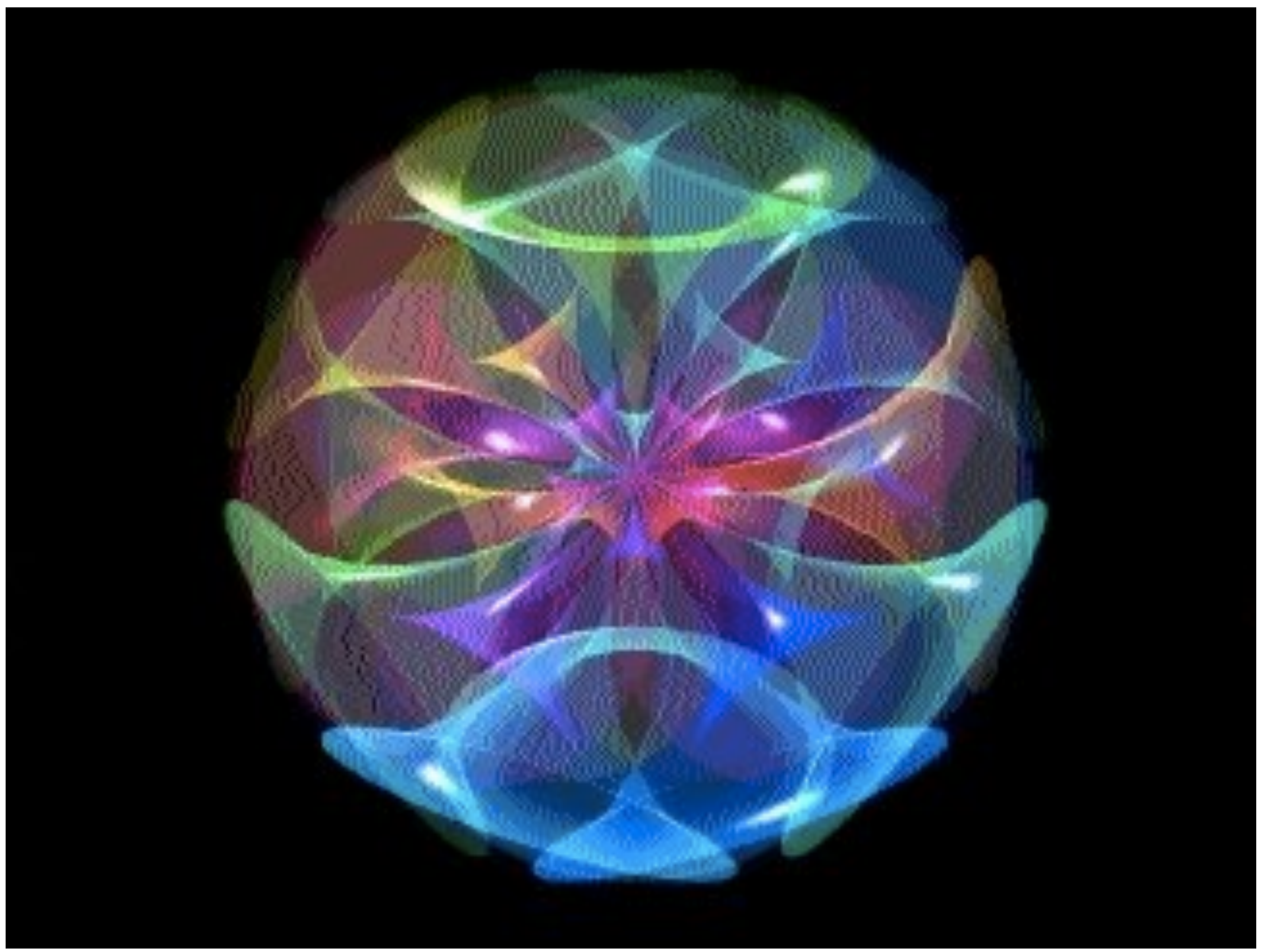


Steven Smale, b. 1930

John Milnor, b. 1931

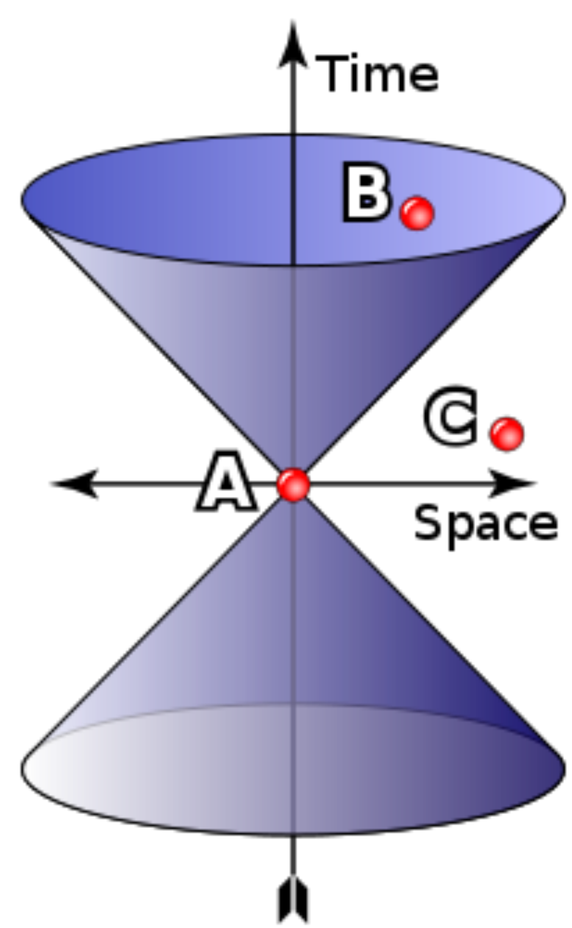
In turns out the geometry is hardest and most mysterious in dimension four!

- The 4-dimensional Poincare conjecture is still wide open!



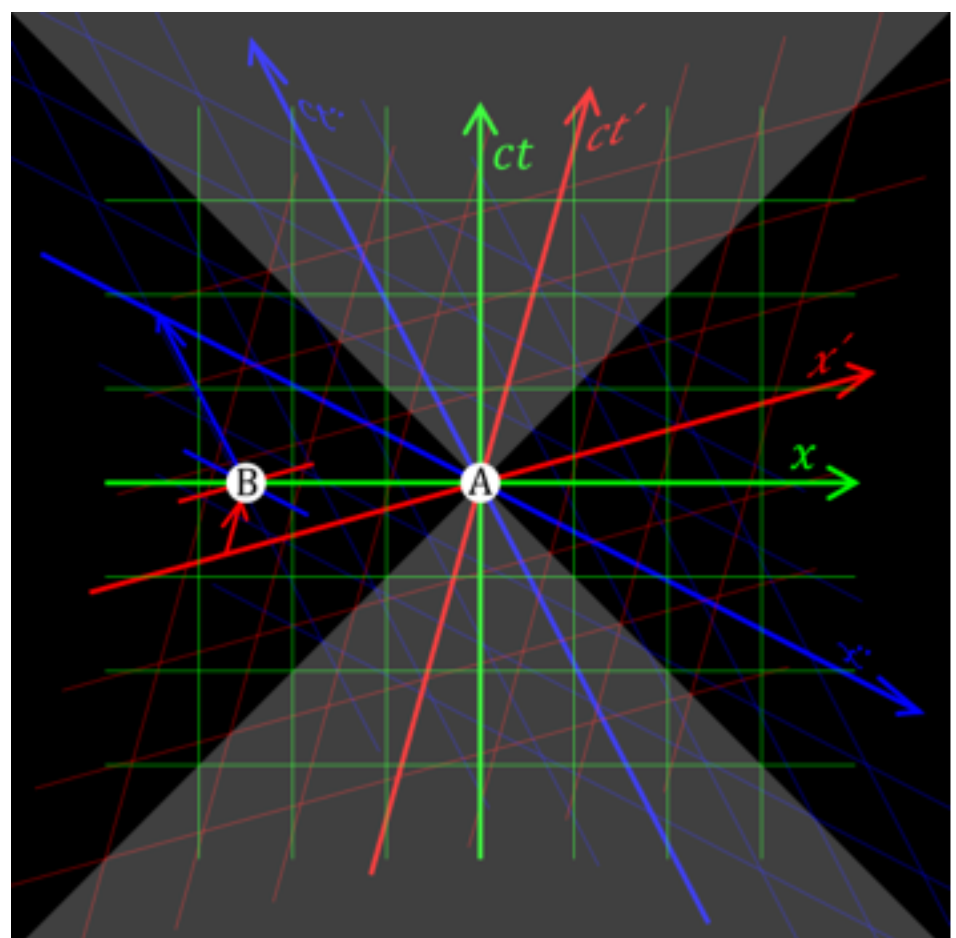
Four dimensional geometry

- According to Einstein's theory of special relatively, we really live in a 4-dimensional "spacetime" in which the notions of space and time cannot truly be separated from each other



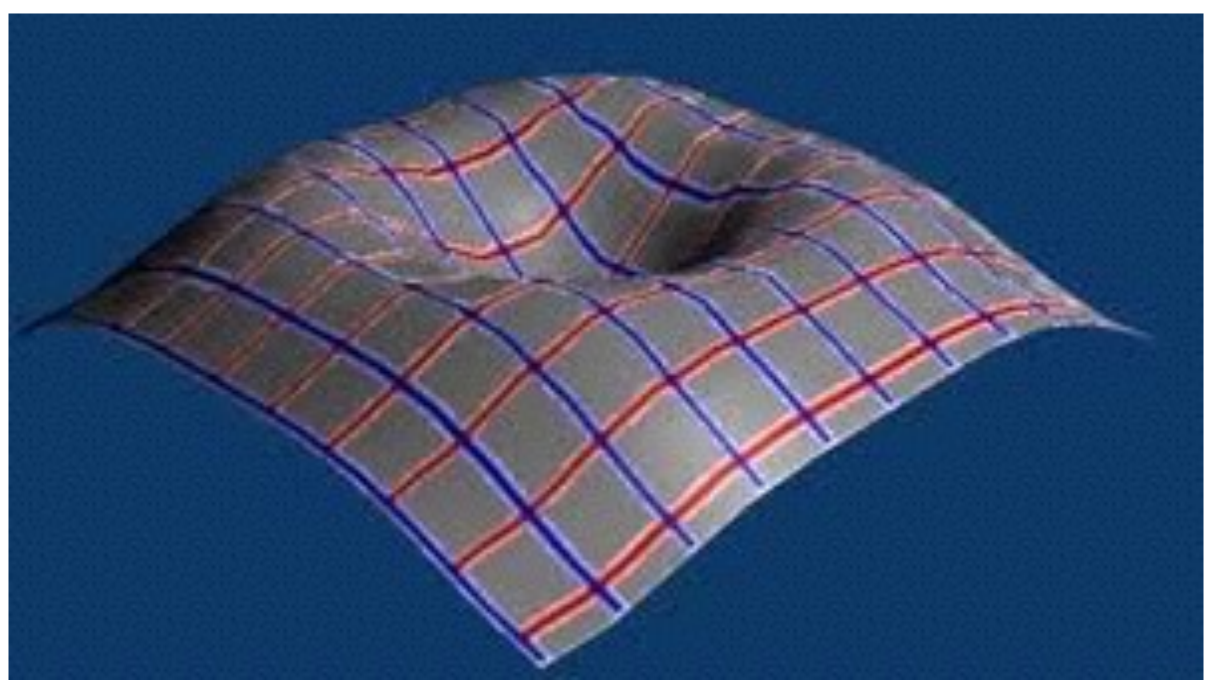
One bizarre consequence:

- The notion of “simultaneity” depends on how fast one is moving!



A theorem in Riemannian geometry:

- The Cartan-Hadamard theorem states that a simply-connected Riemannian manifold with nonpositive curvature must be topologically equivalent to Euclidean space



Our universe

- Current experiments show that our universe appears flat, with a 0.4% margin of error
- Question: does that mean we live in a Euclidean space?

Thanks for listening!